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The Analysis
of Two Simple Composite Structures
Using Mechanical Admittance Methods

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Group 73

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COMPOSITE STRUCTURES USING
MECHANICAL ADMITTANCE METHODS

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ABSTRACT

Mechanical admittance methods are applied to two simple
composite structures. The mechanical admittances of the com-
posite structures are predicted using the mechanical admittances
measured on their constituent substructures. The effects of the
suspension system on the measured admittances are estimated.
Attention is devoted to the technique of analytically assembling
substructures into a composite structure.

Resonant frequencies and mode shapes of the composite
structures are estimated from the predicted admittances. The
effects of modal participation and damping on the estimated mode
shapes and resonant frequencies are discussed.

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$[C]$: a flexibility matrix

$[\overset{\bullet}{C}]$: a diagonal damping matrix

$[J]$: the participation of the $j\underline{^{th}}$ mode in the admittance matrix

$[K]$: a stiffness matrix

$[M]$: a mass matrix

$\{Q\}$: a column matrix of generalized forces

R : deadweight reaction forces

$\{u\}$: a column matrix of generalized displacements

W : a deadweight load

Y : admittance

Z : impedance

a_n : the $n\underline{^{th}}$ modal participation function

d_n : the $n\underline{^{th}}$ modal damping coefficient

$\{f\}$: a column matrix of point forces

i : the imaginary coefficient, $\sqrt{-1}$

K_n : a modal stiffness

m_n : a modal mass

t : time

$\{v\}$: a column matrix of translation velocities

Φ : phase

Ω_n : the $n\underline{^{th}}$ damped resonant frequency

α : the ratio of magnitudes of modal participation

β : the ratio of normalized modal products

ϵ : the participation of all modes but one in the admittance matrix or a displacement

$\mathscr{C}_n$: the nth modal damping function

η : the critical damping coefficient

λ : the length of a shock cord

$\{\phi^n\}$: the nth mode shape

ω : the frequency of excitation

ω_n : the nth resonant frequency

$\{\ \}$: a column matrix

$\lfloor\ \rfloor$: a row matrix

$[\ ]$: a square matrix

$[\ ]^{-1}$: an inverse matrix

$[\ ]^{T}$: matrix transpose

$\nwarrow$: a degree of freedom

$\|\ \|$: the magnitude of a complex number

CHAPTER I

INTRODUCTION

The mechanical admittance[*] method or equivalently the mechanical impedance method is a means of characterizing the dynamic behavior of a structure by observing its dynamic behavior at a finite number of locations. In general, admittance and impedance are complex quanities defined for steady state, sinusoidal motion and are functions of the frequency of excitation. The admittance method is primarily experimental in nature and versatile in its applications.

Mechanical admittance embraces such diverse areas as accoustical transmission in structures to mode shape and resonant frequency determination. Rubin[1] uses admittance methods to reduce levels of engine noise transmitted to passenger compartments on commercial aircraft. Admittance methods are also applied to simple structures to determine resonant frequencies and transmission matrices.[2]

The fundamental difference between an admittance measurement and an impedance measurement is the displacement boundary conditions imposed on the structure during the measurement. For the purposes of this paper impedances are defined as the elements of the inverse of the admittance matrix. For an in depth treatment of the differences between admittance and impedance see reference 3.

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* Admittance is synonymous with the term mobility.

This paper treats the problem of predicting the admittances for a composite structure from the measured admittances of the constituent substructures. In addition, mode shapes and resonant frequencies are estimated from the predicted admittances of the composite structure.

Chapter II deals with the mathematics involved in the admittance prediction technique. Chapter III is a brief description of the total experimental system and the function of the computer programs. Described in Chapter IV is the use of the total system for predicting the admittances, mode shapes and resonant frequencies for two simple composite structures.

CHAPTER II

ANALYTIC BASIS

2.1 The Mechanical Admittance and Mechanical Impedance of a Structure

For most structures of interest their low frequency
dynamic behavior can be modeled by a lumped parameter or
a finite element idealization. Attention is focused on
linear elastic structures undergoing steady state, sinusoidal,
forced excitation. This approach facilitates the derivation
of general expressions for admittance and impedance.

$$\{Q\} = [M]\{\ddot{u}\} + [K]\{u\} \tag{2.1}$$

Equation 2.1 is the general equation of motion for a
linear elastic structure. $\{Q\}$ is a column vector of the
applied forces and moments. $\{u\}$ is a column vector of the
resultant local displacements and rotations of the structure.
The mass matrix, $[M]$, and the stiffness matrix, $[K]$, are
square symmetric matrices of the same order as the number of
degrees of freedom of the structure.
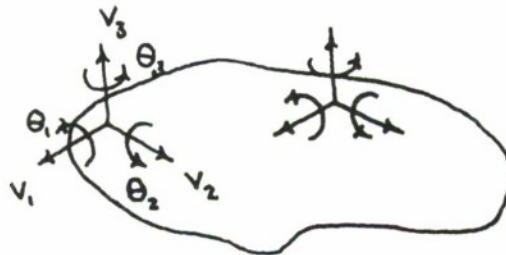
Schematic of the Degrees of Freedom of the Finite
Element Idealization of a Continuous Structure

Figure 2.1

Equation 2.1 is rewritten for steady state, sinusoidal motion.

$$\{\ddot{q}\}e^{i\omega t} = \left[-\omega^2[M] + [K]\right]\{u\}e^{i\omega t} \tag{2.2}$$

$$\{Q\} = \{\ddot{q}\}e^{i\omega t} \tag{2.3}$$

$$\{u\} = \{u\}e^{i\omega t} \tag{2.4}$$

$$\{\ddot{u}\} = -\omega^2\{u\}e^{i\omega t} \tag{2.5}$$

If the mass and stiffness matrices are of order N, then there exist N homogeneous solutions, $\{\phi^n\}$, of equation 2.2

$$\{0\} = \left[-\omega_n^2[M] + [K]\right]\{\phi^n\} \tag{2.6}$$

The homogeneous solutions, $\{\phi^n\}$, exist only for a finite number of non-negative frequencies of motion, ω_n. ω_n is defined as the $n^{\underline{tn}}$ resonant frequency of the structure. The column vector, $\{\phi^n\}$, is defined as the $n^{\underline{th}}$ mode shape of the structure.

In general the mode shapes are orthogonal to each other with respect to both the mass and stiffness matrices. Equations 2.7 and 2.8 are the definitions of orthogonality.

$$\lfloor\phi^j\rfloor[K]\{\phi^k\} = 0. \qquad j \neq k \tag{2.7}$$

$$\lfloor\phi^j\rfloor[M]\{\phi^k\} = 0. \qquad j \neq k \tag{2.8}$$

The resultant motion of the structure for forced excitation is assumed to be a linear combination of the mode shapes.

$$\{u\} = \left[\{\phi^1\}\{\phi^2\}\ldots\{\phi^N\}\right]\{a\} = [\phi]\{a\} \tag{2.9}$$

The elements of the column vector, $\{a\}$, are the modal participation functions which are in general functions of frequency. If the modal participation functions are determined, then the forced response of the structure can be determined.

Substituting equation 2.9 into equation 2.2 and premultiplying both sides of equation 2.2 by $[\phi]^T$, equation 2.10 results.

$$[\phi]^T\{q\}e^{i\omega t} = \left[-\omega^2[\phi]^T[M][\phi] + [\phi]^T[K][\phi]\right]\{a\}e^{i\omega t} \tag{2.10}$$

Invoking the orthogonality of the mode shapes with respect to the mass and stiffness matrices, equations 2.7 and 2.8, the mass and stiffness matrices are transformed into diagonal matrices.

$$[\phi]^T\{q\} = \left[-\omega^2[M^*] + [K^*]\right]\{a\} \tag{2.11}$$

Each modal participation function can now be solved for algebraically.

$$a_n = \frac{\lfloor \phi^n \rfloor \{q\}}{(-\omega^2 m_n + K_n)} \tag{2.12}$$

$$m_n = m^*_{nn} \tag{2.13}$$

$$K_n = K^*_{nn} \tag{2.14}$$

Substituting the expression for the modal participation function, a_n, into equation 2.9 one can construct the forced response of the structure.

$$U_m = u_m e^{i\omega t} = \sum_{n=1}^{N} \frac{\lfloor \phi_n \rfloor \{q\} \phi_m^n e^{i\omega t}}{(-\omega^2 m_n + K_n)} \quad (m=1,2\ldots N) \tag{2.15}$$

For steady atate, sinusoidal motion the resultant velocity at the $m\underline{th}$ degree of freedom of the structure, V_m, can be expressed as a function of the resultant displacement, U_m, and the frequency of excitation, ω .

$$V_m = i\omega u_m e^{i\omega t} = \sum_{n=1}^{N} \frac{\frac{\omega}{m_n} \lfloor \phi^n \rfloor \{q\} \phi_m^n e^{i\omega t}}{i(\omega^2 - \omega_n^2)} \quad (m=1,2\ldots N) \tag{2.16}$$

$$\omega_n^2 = \frac{K_n}{m_m} \tag{2.17}$$

The admittance of a structure, $Y_{m\ell}(\omega)$, is defined as the ratio of the resultant velocity at the $m\underline{th}$ degree of freedom of the structure to the force at the $1\underline{th}$ degree of freedom, assuming all other forces equal to zero.

$$Y_{m\ell}(\omega) = \frac{i\omega u_m e^{i\omega t}}{q_\ell e^{i\omega t}} = \sum_{n=1}^{N} \frac{\frac{\omega}{m_n} \phi_m^n \phi_\ell^n}{i(\omega^2 - \omega_n^2)} \tag{2.18}$$

This expression for admittance[*] is valid for applied moments and rotational velocities. Experimentally, admittances are usually measured only for point forces and trans-

---

[*] If the structure is modeled as a continuum, the expression for admittance is altered slightly. ϕ_m^n is replaced by the $n\underline{th}$ continuous mode shape, ϕ^n, evaluated at the $m\underline{tn}$ degree of freedom of the structure.

lational velocities. For the remainder of the paper admittances are defined for point forces and translational velocities.

The admittances, $Y_{m\ell}(\omega)$, are usually manipulated in matrix form, since admittances are measured on the structure at a finite number of points. Using superposition of velocities, an admittance matrix relating a column vector of point forces, $\{f\}$, to a column vector of translational velocities, $\{v\}$, can be formulated as in equation 2.19.

$$\begin{Bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{Bmatrix} = \begin{bmatrix} Y_{11}(\omega) & Y_{12}(\omega) & \cdots & & Y_{1N}(\omega) \\ Y_{21}(\omega) & Y_{22}(\omega) & & & \vdots \\ \vdots & & & & \vdots \\ Y_{N1}(\omega) & & & & Y_{NN}(\omega) \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{Bmatrix} \qquad (2.19)$$

$$\{v\} = [Y]\{f\} \qquad (2.20)$$

All of the admittances must be evaluated at the same frequency, ω, for the relationship to hold. Elements on the principal diagonal of the admittance matrix are the mutual admittances. All other elements of the admittance matrix are the cross admittances. Using equation 2.18 one can show that the admittance matrix, $[Y]$, is symmetric.

$$Y_{m\ell}(\omega) = \sum_{n=1}^{N} \frac{\frac{\omega}{m_n} \phi_m^n \phi_\ell^n}{i(\omega^2 - \omega_n^2)} = \sum_{n=1}^{N} \frac{\frac{\omega}{m_n} \phi_\ell^n \phi_m^n}{i(\omega^2 - \omega_n^2)} = Y_{\ell m}(\omega) \qquad (2.21)$$

The $p^{\underline{th}}$ column of the admittance matrix, $Y_{jp}(\omega)$ $(j = 1, 2 \ldots N)$,

can be interpreted as the resultant velocities at the N degrees of freedom for a unit sinusoidal force at the pth degree of freedom with no forces applied at the other degrees of freedom. Deleting the pth row and column of the admittance matrix is equivalent to setting the pth force equal to zero.

The inverse of the admittance matrix is defined as the mechanical impedance matrix, $[Z]$.

$$\{f\} = [Y]^{-1}\{\upsilon\} = [Z]\{\upsilon\} \tag{2.22}$$

Since the admittance matrix is symmetric, its inverse, the impedance matrix, is also symmetric. The elements on the principal diagonal of the impedance matrix are the mutual impedances. All other elements of the impedance matrix are the cross impedances.

$$\begin{Bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_N \end{Bmatrix} \quad \begin{bmatrix} Z_{11}(\omega) & Z_{12}(\omega) & \cdot & \cdot & Z_{1N}(\omega) \\ Z_{21}(\omega) & Z_{22}(\omega) & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & \cdot & \\ Z_{N1}(\omega) & \cdot & \cdot & & Z_{NN}(\omega) \end{bmatrix} \begin{Bmatrix} \upsilon_1 \\ \upsilon_2 \\ \cdot \\ \cdot \\ \cdot \\ \upsilon_N \end{Bmatrix} \tag{2.23}$$

The pth column of the impedance matrix, $Z_{jp}(\omega)$ $(j=1,2\ldots N)$, can be interpreted as the resultant forces at the N degrees of freedom for a unit sinusoidal displacement at the pth degree of freedom with no velocity allowed at the other degrees of freedom. Deleting the pth row and column of the impedance

matrix is equivalent to setting the p<u>th</u> velocity equal to zero.

From a practical point of view, making admittance measurements on a structure is easier than making impedance measurements. To make an admittance measurement a force is applied at one degree of freedom and the resultant velocity is monitored at a degree of freedom with no velocity (displacement) boundary conditions imposed on the structure. To make an impedance measurement a velocity must be produced at one degree of freedom while the remaining degrees of freedom are held <u>fixed</u>. The reaction forces at the fixed degrees of freedom as well as the force at the point of excitation must be measured.

The stiffness matrix, $[K]$, and the flexibility matrix, $[C]$, used for static analysis of structures, are analogous to the impedance matrix, $[Z]$, and the admittance matrix, $[Y]$, respectively. Compare equations 2.22 and 2.20 with equations 2.24 and 2.25.

$$\{f^*\} = [K]\{x^*\} \tag{2.24}$$

$$\{x^*\} = [C]\{f^*\} \tag{2.25}$$

$\{f^*\}$ is a column vector of static loads and $\{x^*\}$ is a column vector of static displacements.

2.2 Assembling Substructures into a Composite Structure
Using Mechanical Impedance Notation

Assuming that admittance matrices have been measured
for a number of substructures to be physically assembled
into a composite structute, one can predict the properties
of the composite structure by mathematically assembling the
substructures using impedance notation. If the substructure
admittance matrices are inverted to substructure impedance
matrices, the substructure impedance matrices can be added
to construct the composite structure impedance matrix.

Region d_1 Region c_1 Region c_2 Region d_2

Substructure #1   Substructure #2
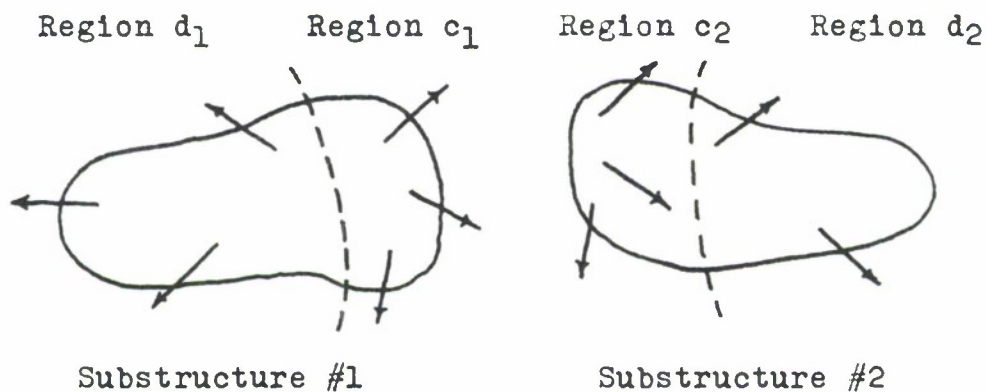
Schematic of Substructures to be

Assembled into a Composite Structure

Figure 2.2

Each substructure in Figure 2.2 is partitioned by a
broken line. Those degrees of freedom that are to be joined
to degrees of freedom on the other substructure are parti-
tioned from the other degrees of freedom in the substructure.

The substructure impedance matrices in equations 2.26

and 2.27 are partitioned in an analogous manner. The elements of each impedance matrix corresponding to degrees of freedom that are to be joined to degrees of freedom on the other substructure are partitioned from the elements of the matrix corresponding to the other degrees of freedom of the substructure. Regions c_1 and c_2 in Figure 2.2 must have the same number of degrees of freedom to insure that $\{v_{c_1}\}$, $\{v_{c_2}\}$, $\{f_{c_1}\}$ and $\{f_{c_2}\}$ all have the same number of elements.

$$\left\{\begin{array}{c} f_{d_1} \\ \hline f_{c_1} \end{array}\right\} = \left[\begin{array}{c|c} Z_{d_1 d_1} & Z_{d_1 c_1} \\ \hline Z_{c_1 d_1} & Z_{c_1 c_1} \end{array}\right] \left\{\begin{array}{c} v_{d_1} \\ \hline v_{c_1} \end{array}\right\} \tag{2.26}$$

$$\left\{\begin{array}{c} f_{c_2} \\ \hline f_{d_2} \end{array}\right\} = \left[\begin{array}{c|c} Z_{c_2 c_2} & Z_{c_2 d_2} \\ \hline Z_{d_2 c_2} & Z_{d_2 d_2} \end{array}\right] \left\{\begin{array}{c} v_{c_2} \\ \hline v_{d_2} \end{array}\right\} \tag{2.27}$$

Equations 2.26 and 2.27 are separated into four matrix equations for the forces in the four regions of Figure 2.2.

$$\{f_{d_1}\} = [Z_{d_1 d_1}]\{v_{d_1}\} + [Z_{d_1 c_1}]\{v_{c_1}\} \tag{2.28}$$

$$\{f_{c_1}\} = [Z_{c_1 d_1}]\{v_{d_1}\} + [Z_{c_1 c_1}]\{v_{c_1}\} \tag{2.29}$$

$$\{f_{c_2}\} = [Z_{c_2 c_2}]\{v_{c_2}\} + [Z_{c_2 d_2}]\{v_{d_2}\} \tag{2.30}$$

$$\{f_{d_2}\} = [Z_{d_2 c_2}]\{v_{c_2}\} + [Z_{d_2 d_2}]\{v_{d_2}\} \tag{2.31}$$

Using equilibrium between internal forces, $\{f_{c_1}\}, \{f_{c_2}\}$ and applied forces, $\{f_c\}$, equations 2.29 and 2.30 can be added to give the applied forces in region c of the composite structure.

$$\{f_c\} - \{f_{c_1}\} - \{f_{c_2}\} = \{0\} \qquad (2.32)$$

$$\{f_c\} = \{f_{c_1}\} + \{f_{c_2}\} = [Z_{c_1 d_1}]\{v_{d_1}\}$$

$$\qquad (2.33)$$

$$+ [Z_{c_1 c_1}]\{v_{c_1}\} + [Z_{c_2 c_2}]\{v_{c_2}\} + [Z_{c_2 d_2}]\{v_{d_2}\}$$

Region d_1 Region c   Region d_2
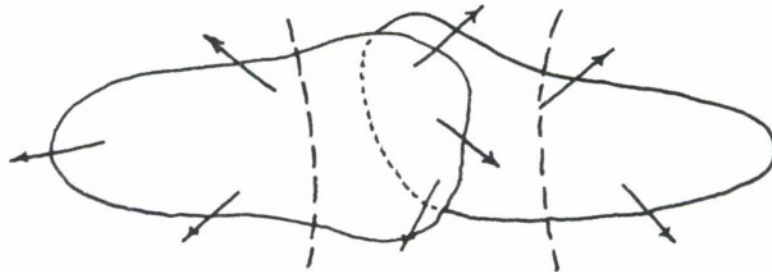
Schematic of Assembled Composite Structure

Figure 2.3

Invoking compatibility of velocities, equation 2.34, at the degrees of freedom common to both substructures in region c, equation 2.33 can be reduced to equation 2.35.

$$\{v_{c_1}\} = \{v_{c_2}\} = \{v_c\} \qquad (2.34)$$

$$\{f_c\} = [Z_{c_1 d_1}]\{v_{d_1}\} + [Z_{c_1 c_1} + Z_{c_2 c_2}]\{v_c\} + [Z_{c_2 d_2}]\{v_{d_2}\} \qquad (2.35)$$

The impedance matrix for the composite structure is

constructed from equations 2.28, 2.31 and 2.35.

$$
\left\{
\begin{array}{c}
f_{d_1} \\ \hline
f_c \\ \hline
f_{d_2}
\end{array}
\right\}
=
\left[
\begin{array}{c|c|c}
Z_{d_1 d_1} & Z_{d_1 c_1} & 0 \\ \hline
Z_{c_1 d_1} & Z_{c_1 c_1} + Z_{c_2 c_2} & Z_{c_2 d_2} \\ \hline
0 & Z_{d_2 c_2} & Z_{d_2 d_2}
\end{array}
\right]
\left\{
\begin{array}{c}
\vartheta_{d_1} \\ \hline
\vartheta_c \\ \hline
\vartheta_{d_2}
\end{array}
\right\}
\qquad (2.36)
$$

Since the measured admittances for the substructures
are valid only for translational velocities, the substructure
impedance matrices are valid only for translational velocities.
Consequently, at the points of attachment of the substruc-
tures one can insure the compatibility of translational
velocities only, equation 2.34. It follows that the attach-
ment points of the first substructure are allowed to rotate
independently from the attachment points of the second sub-
structure. As a result of this limitation in assembling
substructures analytically, composite structures that have
large rotational impedances at the attachment points of
their constituent subsrtuctures will not be accurately
modeled by the composite structure impedance matrix.

Displacement boundary conditions are imposed on the
composite structure by operations on the composite struc-
ture's impedance matrix. One makes the velocity of the pth
degree of freedom zero by deleting the pth row and column
of the impedance matrix. Since impedances are available
just for translational velocities, only translational

velocities of the composite structure can be made zero.

After the displacement boundary conditions are imposed on the composite structure impedance matrix, it is inverted to the composite structure admittance matrix. The resonant frequencies and accompanying mode shapes of the composite structure are estimated from the predicted admittance matrix for the composite structure as shown in section 2.4.

2.3 The Effect of Damping on Resonant Frequencies and Admittances

In section 2.1 the resonant frequencies of a structure are defined as ω_n and the mode shapes are defined as $\{\phi^n\}$. As the frequency of excitation, ω, tends to ω_n, the forced response of the structure tends to infinity where the ratio of the response at one degree of freedom of the structure to the response at another degree of freedom is defined by $\{\phi^n\}$. The property of damping exhibited in most real structures changes the definition of resonant frequencies. Consequently, damping affects the magnitude and phase of an admittance at or near a resonant frequency of the structure.

To facilitate the discussion of damping it is assumed that the damping present in the structure is modal damping. Each mode is assumed to exhibit damping independently of the damping of other modes.

This form of damping is represented mathematically by introducing a diagonal damping matrix into equation 2.11.

$$\{q\} e^{i\omega t} = \left[-\omega^2 [M^*] + i\omega [C^*] + [K^*] \right] \{a\} e^{i\omega t} \qquad (2.37)$$

The modal participation functions, a_n, are solved for in the same manner as in section 2.1.

$$a_n = \frac{\frac{1}{m_n} \lfloor \phi^n \rfloor \{q\}}{i\omega \ell_n(\omega) + (\omega_n^2 - \omega^2)} \qquad (2.38)$$

$$\mathcal{C}_n(\omega) = \frac{C^*_{nn}(\omega)}{m_{nn}} \qquad (2.39)$$

$\mathcal{C}_n(\omega)$ is the modal damping coefficient. For viscous damping $\mathcal{C}_n(\omega)$ is $d_n\omega_n$. For hysteretic damping $\mathcal{C}_n(\omega)$ is of the form shown in equation 2.40.

$$\mathcal{C}_n(\omega) = \frac{d_n\omega_n^2}{\omega} \qquad (2.40)$$

The admittance of a structure with modal damping is given in equation 2.41.

$$Y_{m\ell}(\omega) = \sum_{n=1}^{N} \frac{\frac{\omega}{m_n}\phi_m^n \phi_\ell^n}{\mathcal{C}_n(\omega)\omega + i(\omega^2 - \omega_n^2)} = \sum_{n=1}^{N} \frac{\frac{\omega}{m_n}\phi_\ell^n \phi_m^n}{\mathcal{C}_n(\omega)\omega + i(\omega^2 - \omega_n^2)} = Y_{\ell m}(\omega) \quad (2.41)$$

The admittance matrix for a structure with modal damping is symmetric. Consequently, the impedance matrix for a structure with modal damping is also symmetric.

The poles of equation 2.41 are the damped resonant frequencies of the structure, Ω_n, and are, in general, complex numbers.

$$\mathcal{C}_n(\Omega_n)\Omega_n + i(\Omega_n^2 - \omega_n^2) = 0. \qquad (2.42)$$

The damped resonant frequencies can be solved for explicitly for viscous damping, equation 2.43, and hysteretic damping, equation 2.44.

$$\Omega_n = \omega_n\left(1 - \frac{d_n^2}{4}\right)^{1/2} + i\omega_n\frac{d_n}{2} \qquad \mathcal{C}_n(\omega) = d_n\omega_n \qquad (2.43)$$

$$\Omega_n = \omega_n\left(1 + id_n\right)^{1/2} \qquad \mathcal{C}_n(\omega) = \frac{d_n\omega_n^2}{\omega} \qquad (2.44)$$

The damping coefficient, d_n , is much less than one for low frequencies of excitation in most structures. For hysteretic and viscous damping d_n^{-1} is the Q_n^* factor or the resonant quality of the nth mode. For the remainder of the chapter Q_n^* is assumed to be 50.

If the frequency of excitation is near the jth undamped resonant frequency, ω_j , the jth term in equation 2.41 is most representative of the structure's admittance.

$$Y_{m\ell}(\omega) = \frac{\frac{\omega}{m_j}\, \phi_m^j\, \phi_\ell^j}{\mathcal{C}_j(\omega)\omega + i(\omega^2 - \omega_j^2)} + \mathcal{E}_{m\ell} \qquad (2.45)$$

$$\mathcal{E}_{m\ell} = \sum_{\substack{n=1 \\ n \neq j}}^{N} \frac{\frac{\omega}{m_n}\, \phi_m^n\, \phi_\ell^n}{\mathcal{C}_n(\omega)\omega + i(\omega^2 - \omega_n^2)} \qquad (2.46)$$

$\mathcal{E}_{m\ell}$ represents the participation of all modes but the jth for $\omega \approx \omega_j$. The admittance matrix for the structure excited near the jth undamped resonant frequency is given in equation 2.47.

$$\begin{Bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{Bmatrix} = \frac{\frac{\omega}{m_j}}{\mathcal{C}_j(\omega)\omega + i(\omega^2 - \omega_j^2)} \begin{bmatrix} \phi_1^j \phi_1^j & \phi_1^j \phi_2^j & \cdots & \phi_1^j \phi_N^j \\ \phi_2^j \phi_1^j & \phi_2^j \phi_2^j & & \\ \vdots & & & \\ \phi_N^j \phi_1^j & & & \phi_N^j \phi_N^j \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{Bmatrix}$$

$$+ \begin{bmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} & & & \mathcal{E}_{1N} \\ \mathcal{E}_{21} & \mathcal{E}_{22} & & & \\ \vdots & & & & \\ \mathcal{E}_{N1} & & & & \mathcal{E}_{NN} \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{Bmatrix} \qquad (2.47)$$

$$\{v\} = [Y]\{f\} = [J]\{f\} + [\epsilon]\{f\} \qquad (2.48)$$

If all $\epsilon_{m\ell}$ are zero for $\omega \approx \omega_j$, then the matrix, $[J]$, is equal to the admittance matrix. All of the elements of $[J]$ are either in phase, 0 radians, or out of phase, π radians, with respect to each other for $\omega \approx \omega_j$. All the elements of $[J]$ have a local maximum at one frequency of excitation. This frequency is the real part of the damped resonant frequency defined in equation 2.42.

In general the participation of all modes except the $j\underline{\text{th}}$, $\epsilon_{m\ell}$, is not zero for $\omega \approx \omega_j$. It follows that the elements of the admittance matrix, $J_{m\ell} + \epsilon_{m\ell}$, will not have their local maxima at exactly the same frequency for $\omega \approx \omega_j$. In addition the elements of the admittance matrix are not exactly in or out of phase with respect to each other for any frequency of excitation.

2.4 Estimating Resonant Frequencies and Mode Shapes for Composite Structures from Predicted Admittance Data

The predicted admittances of a structure are the normalized levels of the forced response. The levels of the forced response are normalized by the magnitudes of the applied forces.

Equation 2.47 is representative of the behavior of an admittance matrix at or near the jth undamped resonant frequency of the structure.

$$
\begin{Bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{Bmatrix} = \frac{\frac{\omega}{m_j}}{\varrho_j(\omega)\omega + i(\omega^2 - \omega_j^2)} \begin{bmatrix} \phi_1^j \phi_1^j & \phi_1^j \phi_2^j & \cdots & \phi_1^j \phi_N^j \\ \phi_2^j \phi_1^j & \phi_2^j \phi_2^j & & \vdots \\ \vdots & & \ddots & \\ \phi_N^j \phi_1^j & & \cdots & \phi_N^j \phi_N^j \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{Bmatrix}
$$

$$
+ \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \cdots & \epsilon_{1N} \\ \epsilon_{21} & \epsilon_{22} & & \vdots \\ \vdots & & \ddots & \\ \epsilon_{N1} & & \cdots & \epsilon_{NN} \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{Bmatrix} \tag{2.47}
$$

$$
\{v\} = [Y]\{f\} = [J]\{f\} + [\epsilon]\{f\} \tag{2.48}
$$

If the participation of all modes except the jth, $\epsilon_{m\ell}$, is very small compared to the participation of the jth mode, then one can predict the jth resonant frequency and mode shape from the admittance matrix. The frequency at which the elements of the admittance matrix have their j^{th} local maximum is the jth resonant frequency. The rows and columns

of the admittance matrix are proportional to the $j\underline{th}$ mode shape, $\left\{\phi^j\right\}$, for $\omega \approx \omega_j$.

In general there are two cases where the participation of all modes except the $j\underline{th}$ is not small compared to the participation of the $j\underline{th}$ mode for $\omega \approx \omega_j$. First, if an admittance is predicted for a degree of freedom at or near a node of the $j\underline{th}$ mode shape, that admittance is more representative of the participation of modes other than the $j\underline{th}$. Second, if two resonant frequencies of the structure are close together, the admittance matrix is representative of the participation of both modes at either resonant frequency.

If an element of the admittance matrix corresponds to a degree of freedom near a node of the $j\underline{th}$ mode shape and is dominated by the participation of modes other than the $j\underline{th}$, then that element will not have a well defined maximum in magnitude in the region of the $j\underline{th}$ resonant frequency.

$$\mathcal{E}_{m\ell} = \sum_{\substack{n=1 \\ n \neq j}}^{N} \frac{\frac{\omega}{m_n} \phi_m^n \phi_\ell^n}{\mathcal{C}_{\ell n}(\omega)\omega + i(\omega^2 - \omega_n^2)} \tag{2.46}$$

The expression for the participation of all modes but the $j\underline{th}$, $\mathcal{E}_{m\ell}$, does not have any poles near ω_j. Therefore, there is no local maximum in magnitude of $\mathcal{E}_{m\ell}$ for $\omega \approx \omega_j$.

If two resonant frequencies of the structure are close together, then the admittance can be approximated by the participation of only those two modes for frequencies of excitation near the two resonant frequencies.

$$Y_{m\ell}(\omega) = \frac{\frac{\omega}{m_j}\,\phi_m^j\,\phi_\ell^j}{\mathcal{C}_j(\omega)\omega + i\,(\omega^2 - \omega_j^2)} \quad + \quad \frac{\frac{\omega}{m_p}\,\phi_m^p\,\phi_\ell^p}{\mathcal{C}_p(\omega)\omega + i\,(\omega^2 - \omega_p^2)} \tag{2.49}$$

Evaluating the admittance at ω_j one can solve for values of $\frac{\omega_p}{\omega_j}$ such that the magnitude of the participation of the pth mode is α of the participation of the jth mode at the jth resonant frequency.

$$\alpha \left\| \frac{\frac{\omega_j}{m_j}\,\phi_m^j\,\phi_\ell^j}{\mathcal{C}_j(\omega_j)\omega_j + i\,(\omega_j^2 - \omega_j^2)} \right\| = \left\| \frac{\frac{\omega_j}{m_p}\,\phi_m^p\,\phi_\ell^p}{\mathcal{C}_p(\omega_j)\omega_j + i\,(\omega_j^2 - \omega_p^2)} \right\| \tag{2.50}$$

The damping for both modes is assumed to be viscous.

$$\mathcal{C}_j(\omega) = d_j\omega_j = 2\eta_j\omega_j \tag{2.51}$$

$$\mathcal{C}_p(\omega) = d_p\omega_p = 2\eta_p\omega_p \tag{2.52}$$

η is the ratio of the damping of the mode to critical damping for that mode. The ratio of the normalized modal products for the pth and j^{th} modes is β.

$$\beta = \frac{\frac{1}{m_p}\,\phi_m^p\,\phi_\ell^p}{\frac{1}{m_j}\,\phi_m^j\,\phi_\ell^j} \tag{2.53}$$

Substituting equations 2.51, 2.52 and 2.53 into equation 2.50 one can solve for values of $\frac{\omega_p}{\omega_j}$.

$$\lambda^2 - (2 - 4\eta_p^2)\lambda + 1 - 4\frac{\eta_j^2\beta^2}{\alpha^2} = 0. \tag{2.54}$$

$$\lambda = \frac{\omega_p^2}{\omega_j^2} \tag{2.55}$$
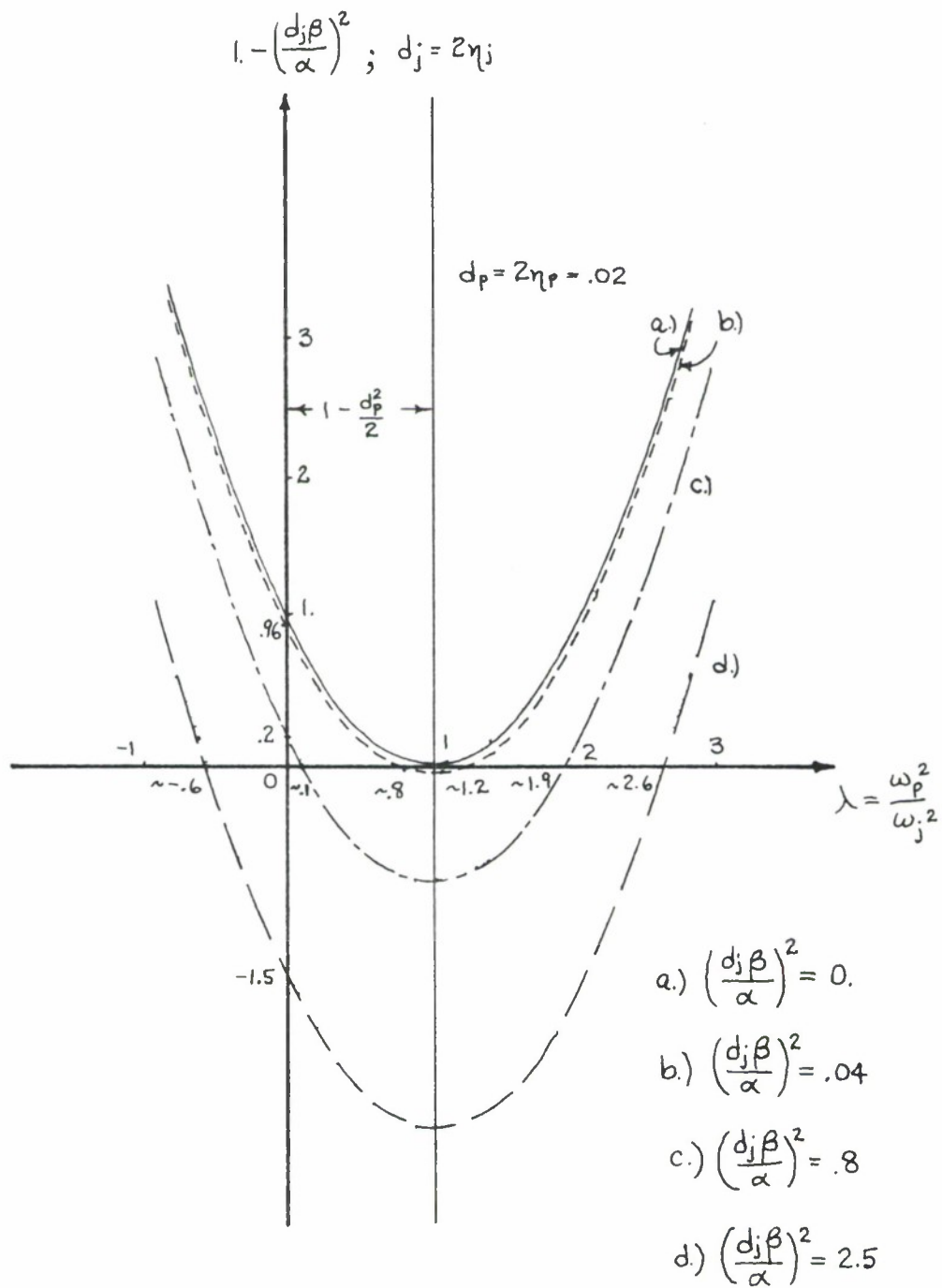
Plots of Equation 2.54 for Different
Values of the Parameters d_n, α, and β

Figure 2.4

Figure 2.4 shows plots of equation 2.54 for different values of the parameters, d_n, α and β . Assuming the modal damping coefficients d_n are .02, the ratio of the normalized modal products β is 1.0 and α is .1, the roots of equation 2.54 are given in equations 2.56 and 2.57.

$$\lambda = \frac{\omega_p^2}{\omega_j^2} \approx 1.2 \tag{2.56}$$

$$\lambda = \frac{\omega_p^2}{\omega_j^2} \approx .8 \tag{2.57}$$

Thus, if $\frac{\omega_p^2}{\omega_j^2}$ is greater than .8 and less than 1.2, the participation of the p$\underline{th}$ mode at the j$\underline{th}$ resonant frequency is greater than 10%. If $\frac{\omega_p^2}{\omega_j^2}$ is less than .8 or greater than 1.2, the participation of the p$\underline{th}$ mode at the j$\underline{th}$ resonant frequency is less than 10%.

The participation of the p$\underline{th}$ mode affects the phase of the admittance at the j$\underline{th}$ resonant frequency. The two modal participation terms in equation 2.49 are treated as vectors in the complex plane.

$$Y_{m\ell}(\omega) = \frac{\frac{\omega}{m_j} \phi_m^j \phi_\ell^j}{\zeta_j(\omega)\omega + i(\omega^2 - \omega_j^2)} + \frac{\frac{\omega}{m_p} \phi_m^p \phi_\ell^p}{\zeta_p(\omega)\omega + i(\omega^2 - \omega_p^2)} \tag{2.49}$$
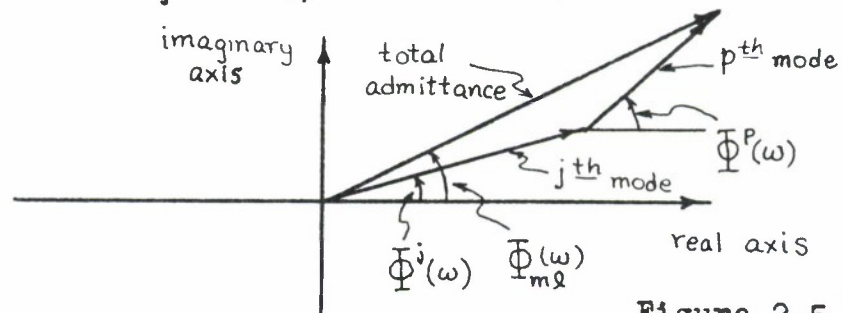
Figure 2.5

The Modal Participation Vectors of the P$\underline{th}$ and J$\underline{th}$ Modes in the Complex Plane

The phases of the two modal participation vectors evaluated at ω_j are given in equations 2.58 through 2.61.

$$\Phi^j(\omega_j) = -\tan^{-1}\left\{\frac{\omega_j^2 - \omega_j^2}{\mathcal{C}_j(\omega_j)\omega_j}\right\} = 0. \qquad \phi_m^j \phi_\ell^j > 0 \qquad (2.58)$$

$$\Phi^j(\omega_j) = \pi - \tan^{-1}\left\{\frac{\omega_j^2 - \omega_j^2}{\mathcal{C}_j(\omega_j)\omega_j}\right\} = \pi \quad \phi_m^j \phi_\ell^j < 0 \qquad (2.59)$$

$$\Phi^p(\omega_j) = -\tan^{-1}\left\{\frac{\omega_j^2 - \omega_p^2}{\mathcal{C}_p(\omega_j)\omega_j}\right\} \qquad \phi_m^p \phi_\ell^p > 0 \qquad (2.60)$$

$$\Phi^p(\omega_j) = \pi - \tan^{-1}\left\{\frac{\omega_j^2 - \omega_p^2}{\mathcal{C}_p(\omega_j)\omega_j}\right\} \qquad \phi_m^p \phi_\ell^p < 0 \qquad (2.61)$$

If M_j is the magnitude of the participation of the jth mode and M_p is the magnitude of the participation of the pth mode, the phase of the admittance at the jth resonant frequency is given in equations 2.62 and 2.63.

$$\Phi_{m\ell}(\omega_j) = \tan^{-1}\left\{\frac{M_p \sin \Phi^p(\omega_j)}{M_j + M_p \cos \Phi^p(\omega_j)}\right\} \qquad \Phi^j(\omega_j) = 0 \qquad (2.62)$$

$$\Phi_{m\ell}(\omega_j) = \pi + \tan^{-1}\left\{\frac{-M_p \sin \Phi^p(\omega_j)}{-M_j + M_p \cos \Phi^p(\omega_j)}\right\} \qquad \Phi^j(\omega_j) = \pi \qquad (2.63)$$

The participation of the pth mode affects the phase of the admittance most when $\Phi^p(\omega_j) = \pm\frac{\pi}{2}$.

$$\Phi_{m\ell}(\omega_j) = \tan^{-1}\left\{\pm\frac{M_p}{M_j}\right\} = \tan^{-1}\left\{\pm\alpha\right\} \approx \pm\alpha \qquad \Phi^j(\omega_j) = 0 \qquad (2.64)$$

$$\Phi_{m\ell}(\omega_j) = \pi + \tan^{-1}\left\{\pm\frac{M_p}{M_j}\right\} = \tan^{-1}\left\{\pm\alpha\right\} \approx \pi \pm \alpha \qquad (2.65)$$
$$\Phi^j(\omega_j) = \pi$$

The effect of the pth mode on the phase of the admittance at the jth resonant frequency is at most α radians. α is defined in equation 2.50.

Assuming that the jth resonant frequency is not close to other resonant frequencies of the structure, the jth mode shape is calculated from the maximum values of the magnitudes for the admittances near the jth resonant frequency. An element of the admittance matrix that does not have a local maximum in magnitude for $\omega \approx \omega_j$ and is dominated by the participation of modes other than the jth is not used to calculate the jth mode shape.

The signs of the components of the mode shape vector, $\{\phi^{'}\}$, are determined by the relative phases of the admittances at the jth resonant frequency. Unfortunately, the admittance data is available only for a finite number of frequencies which may not coincide with the exact resonant frequencies of the composite structure. However, as long as the participation of modes other than the jth is small for $\omega \approx \omega_j$, the relative phases of the admittances at the jth resonant frequency are approximately the same as their relative phases near the jth resonant frequency.

The matrix $[J]$ is approximately equal to the admittance matrix at the jth resonant frequency if the participation of modes other than the jth is small.[*] Equations 2.66 and 2.67 give the phases of elements of the matrix $[J]$.

---

[*] See equation 2.48.

$$\Phi_{m\ell}(\omega) = -\tan^{-1}\left\{\frac{\omega^2 - \omega_j^2}{C_{ij}(\omega)\omega}\right\} \qquad \phi_m^j\,\phi_\ell^j > 0 \qquad (2.66)$$

$$\Phi_{m\ell}(\omega) = \pi - \tan^{-1}\left\{\frac{\omega^2 - \omega_j^2}{C_{ij}(\omega)\omega}\right\} \qquad \phi_m^j\,\phi_\ell^i < 0 \qquad (2.67)$$

The relative phases of the elements of the matrix $[J]$ are
0 or π radians for all frequencies of excitation, ω . These
relative phases determine the signs of the components of the
$j^{\underline{th}}$ mode shape vector, $\{\phi^j\}$ and are approximately equal to
the relative phases of the admittances at the $j^{\underline{th}}$ resonant
frequency of the structure.

CHAPTER III

The Mechanical Admittance Measurement and Prediction System

The primary function of the measurement and prediction system is to predict the admittances of a composite structure from admittance measurements made on the constituent substructures. The admittance measurements for the substructures are made in analog form and then converted to digital form to construct the substructure admittance matrices. The substructure admittance matrices are used to calculate the composite structure admittance matrix.

The admittances of a substructure are constructed from force and acceleration measurements made on the substructure. The force and acceleration measured are resultants of a force input to the structure generated by an electromagnetic shaker. The resultant force imparted to the substructure is measured by a very stiff, lightweight strain gauge mounted between the shaker and the substructure. The resultant acceleration is measured by an accelerometer mounted behind the strain gauge or at some location on the substructure.

The force input to the structure generated by the shaker is proportional to the deflection of the strain gauge. Because the alloy in which the strain gauge is embedded is stiff and lightweight, the acceleration measured behind the strain gauge is the local acceleration of the substructure

The Mechanical Admittance Measurement and Prediction System

Figure 3.1
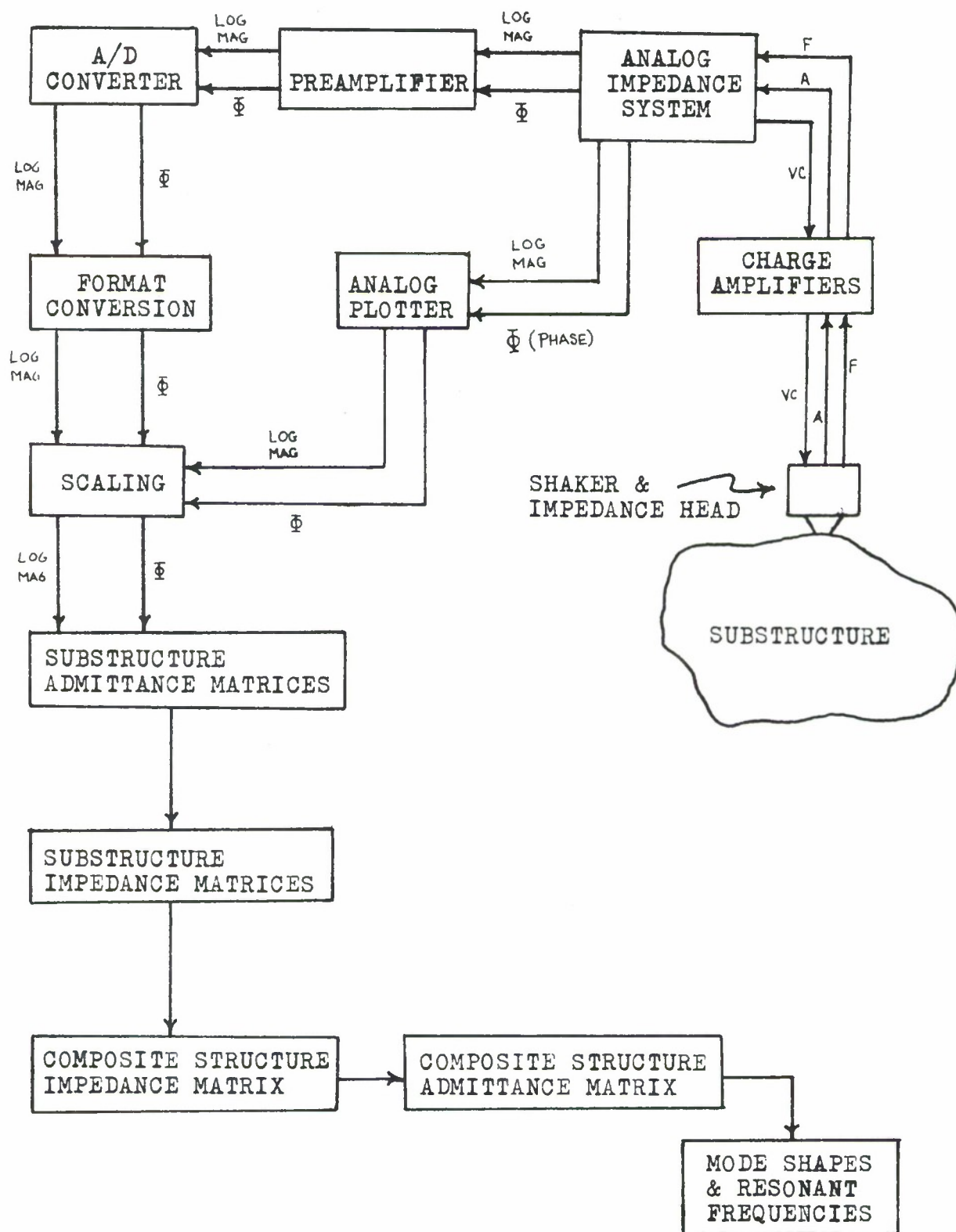
in front of the strain gauge except at very high frequencies of excitation.[4]
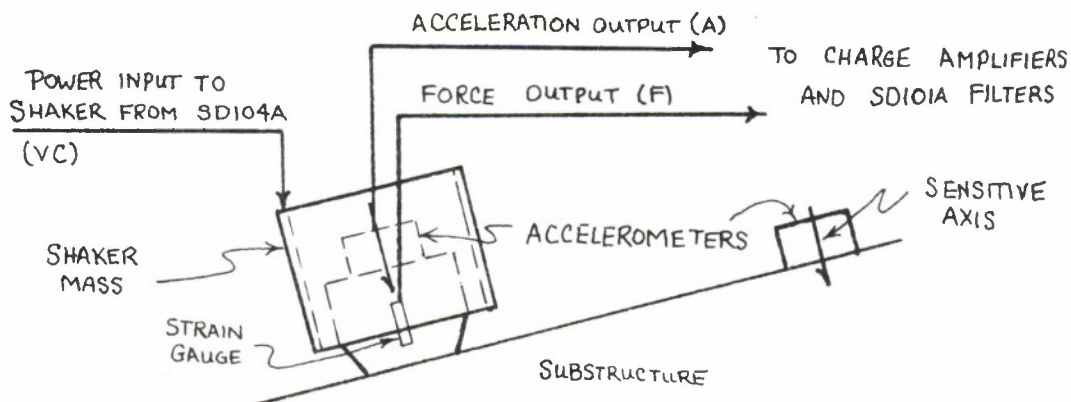
Schematic of Shaker and Impedance Head

Figure 3.2

A strain gauge with an accelerometer mounted behind it as in Figure 3.2 is referred to as an "impedance head". The device used in the following experiments is a Z602 impedance head made by Wilcoxon Research. The shaker is a F3 "wrap around" shaker also made by Wilcoxon Research.[*]

The admittances of a substructure are measured over a finite frequency range by varying the frequency of the input to the shaker from the SD104A sweep oscillator. The resultant force and acceleration signals are amplified and put into the analog impedance equipment. These signals are band pass filtered before they are used to calculate the $\log_{10}$(magnitude) and phase of the measured admittance.

Each SD101A filter is made of many band pass filters.

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* The specifications for these instruments appear in reference 5.

Schematic of the SD1002B-33 Automatic Mechanical Impedance

System[*] in the Mechanical Admittance Mode

Figure 3.3

---
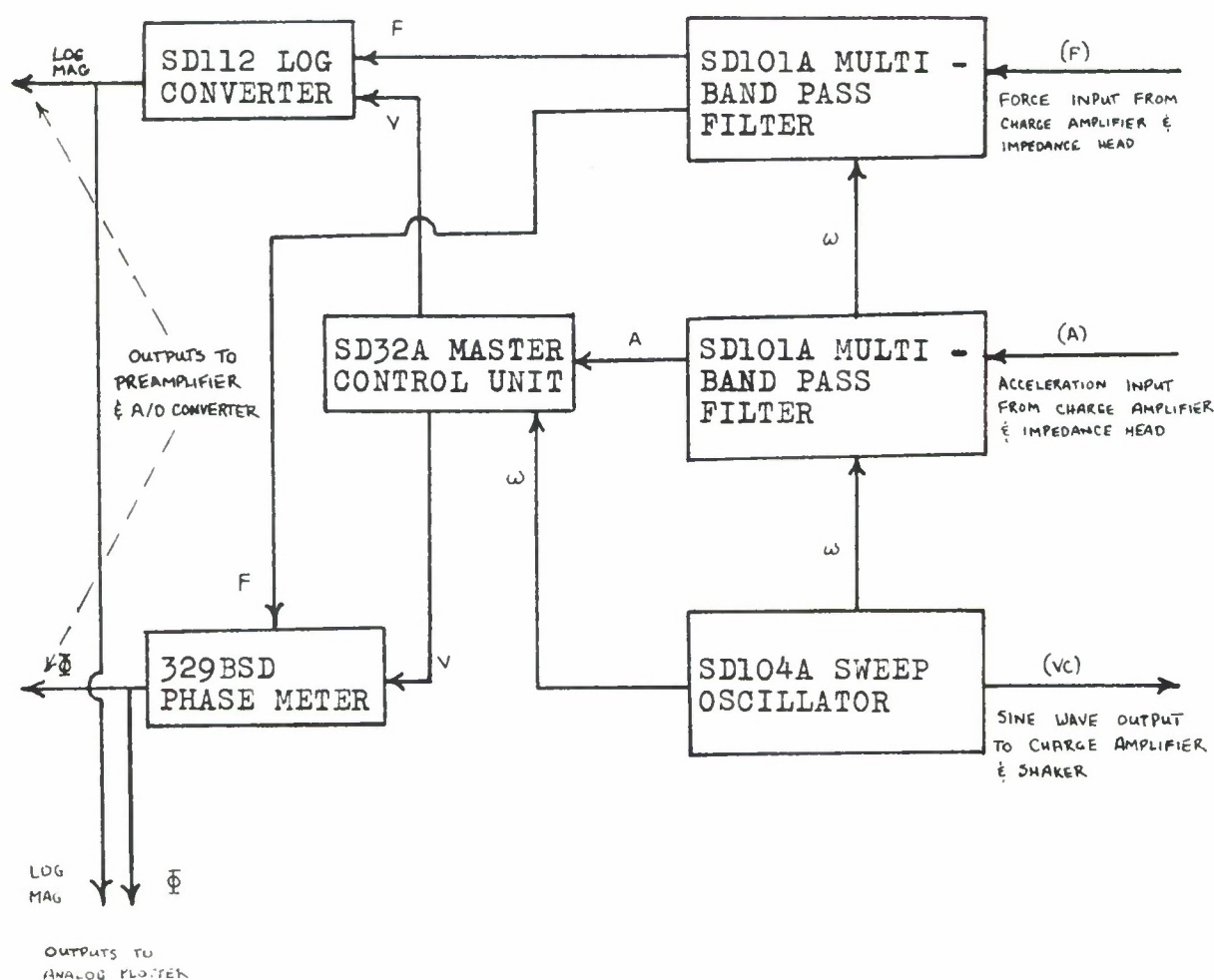
[*] For a complete description of the SD1002B-33 Automatic

Mechanical Impedance System, see reference 7.

Each band pass filter has a unique center frequency and is
active over a finite frequency range that includes its
center frequency. The active ranges of the band pass fil-
ters cover a broad frequency range, 5 Hz. to 10 kilo-Hz.,
and they do not overlap.
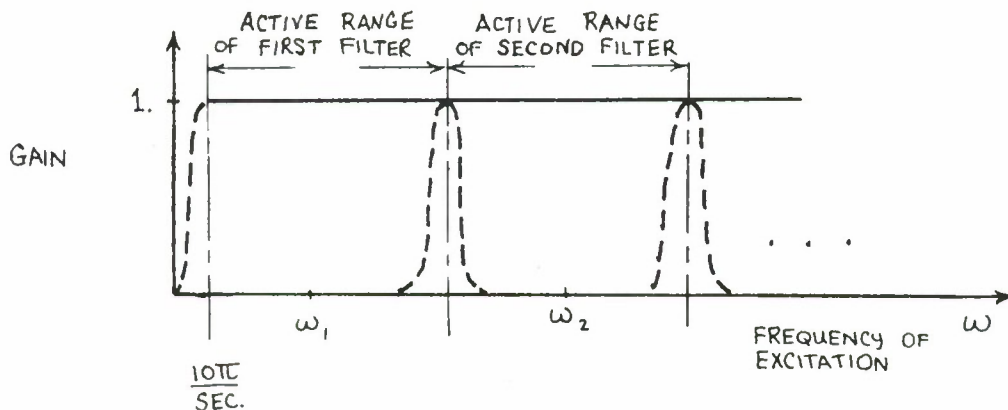
Schematic of the Transfer Function of a
SD101A Filter in the Frequency Domain

Figure 3.4

Only one band pass filter is active for a given fre-
quency of excitation generated by the SD104A sweep oscil-
lator. The resultant force and acceleration signals
generated by the impedance head should have the same fre-
quency as the frequency of excitation for the shaker
generated by the SD104A. If other frequencies are present
in the force and acceleration signals outside the band of
the active filter, they are eliminated from the signals.
This filtering is done to make more accurate phase and mag-
nitude calculations for the admittance.

The filtered acceleration signal is divided by the
frequency of excitation and shifted in phase by $-\frac{\pi}{2}$ radians

to yield the velocity of the substructure. The velocity
signal and the filtered force signal are used to calculate
the admittance of the structure. The SD112 log converter
generates a voltage proportional to the $\log_{10}$(magnitude) of
the admittance. The 329BSD phase meter generates a voltage
proportional to the phase of the admittance.

The voltages proportional to phase and $\log_{10}$(magnitude)
are plotted on a 136AR Hewlett Packard X Y_1 Y_2 Recorder.
These voltages are also preamplified and put into a Radiation
analog-to-digital (A/D) converter.

The A/D converter has one analog input channel and
generates 12 bit digital words from the input voltage. The
input voltage range is -2 volts to +2 volts. With a twelve
bit register the A/D converter can generate 4096 different
numbers corresponding to the four volt input range or roughly
one number for every millivolt input.

The outputs from the analog impedance system are
typically 50 millivolts. To achieve any accuracy in the
digital data these signals must be preamplified before they
are put into the A/D converter. After making a preliminary
frequency sweep to establish the maximum and minimum values
of the voltage output from the impedance system, the bias
and gain are adjusted on the preamplifier to take full
advantage of the A/D converter's four volt input range.

---

* For a complete description of the Radiation A/D converter
see reference 6.

The SD1002B-33 Automatic Mechanical
Impedance System and Charge Amplifiers

Figure 3.5

A second frequency sweep is made to sample and convert the amplified analog signal from the impedance system, and the digital data is stored on magnetic tape.

Because the A/D converter has only one input channel the $\log_{10}$(magnitude) and the phase of the admittance are converted on two seperate frequency sweeps.[*] The digital data is converted in format from 12 bit words to 32 bit words that the IBM 360 computer can understand. Each 12 bit word is padded with 20 extra bits without changing the value of the number in the word by the computer program CONVERT.

The digital data is scaled from the analog plots made on the Hewlett Packard recorder. Two values of the phase or magnitude from a plot, P_1 and P_2, are compared to two values of the digital data, D_1 and D_2, for corresponding frequencies of excitation. The gain and bias introduced into the digital data by the preamplifier are calculated.

$$\text{GAIN} = \frac{(D_2 - D_1)}{(P_2 - P_1)} \tag{3.1}$$

$$\text{BIAS} = \frac{D_1(P_2 - P_1)}{(D_2 - D_1)} - P_1 \tag{3.2}$$

---

[*] If a two track tape recorder is available it is better to record phase and $\log_{10}$(magnitude) on the same frequency sweep and then to put the analog signals into the A/D converter one track at a time.

All of the digital data corresponding to the analog plot is
scaled using the calculated gain and bias. D_m is the
unscaled value of the digital data for the m$^{\underline{th}}$ frequency of
excitation. S_m is the scaled value of D_m.

$$S_M = -(\text{BIAS}) + (\text{GAIN})^{-1} D_M = P_1 - \frac{P_2 - P_1}{D_2 - D_1}(D_M - D_1) \qquad (3.3)$$

This scaling process must be done twice for each admittance,
once for phase and once for magnitude. All scaling is done
by the computer program SCALE.

The scaled data is used to construct the substructure
admittance matrix for the sampled frequencies of excitation.
After all of the substructure admittance matrices are con-
structed they are inverted* to substructure impedance matri-
ces. The substructure impedance matrices are assembled into
the composite structure impedance matrix. All matrix inver-
sion and assembling is done by the computer program ADMIT.

The mode shapes and resonant frequencies of the com-
posite structure are estimated from the admittances of the
composite structure as shown in section 2.4.

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* The method of matrix inversion is described in Appendix A.

CHAPTER IV

EXPERIMENTAL RESULTS

4.1 Introduction to Experimental Results

The admittance measurement and prediction system is applied to two composite structures. The first composite structure is an 80-inch free rod assembled from two 40-inch free rods. The second composite structure is two 32-inch beams pinned together.
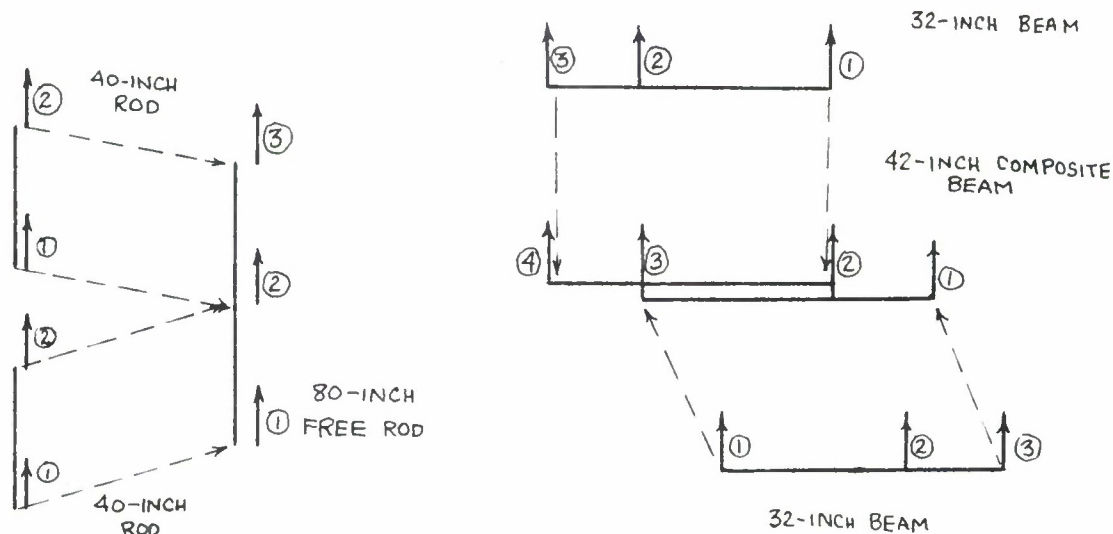
Schematic of Assembling Two Composite Structures

from Their Constituent Substructures

Figure 4.1

Admittance measurements are made only on the substructures. The "predicted" admittances are the admittances of the composite structure predicted by the computer program ADMIT. The "calculated" admittances are generated by analytical modal models of the structures and are used only for comparison with the "measured" and "predicted" admittances.

Measuring the Cross Admittance
of a 40-Inch Steel Rod

Figure 4.2

4.2 The Effects of the Suspension System, Impedance Head and Shaker on the Measured Admittances of a 40-Inch Free Rod

A rubber band suspension system shown in Figure 4.2 is used to simulate a free rod for frequencies of excitation, ω, well above the resonant frequency of the suspension system. The resonant frequency of the rod and rubber band ensemble is 1.2 c.p.s., which is well below the lowest frequency for which admittance measurements are made on the rod ($\frac{\omega}{2\pi} \geq 50$ c.p.s.).

In order to estimate the effect of the rubber band on the measured admittances, the rod and rubber band are modeled as a two degree of freedom, lumped parameter system. See Figure 4.3. Because the model has only two degrees of freedom it will be representative of the system for frequencies of excitation up to, and including, the second resonant frequency of the rod and rubber band ensemble. However, this two-degree of freedom model will give some indication of the effects of the rubber band suspension system on the measured admittances of the rod.

k_B - the stiffness of the rubber band

c_B - " damping " " " "

k_R - the stiffness of the rod

m - the lumped masses of the rod
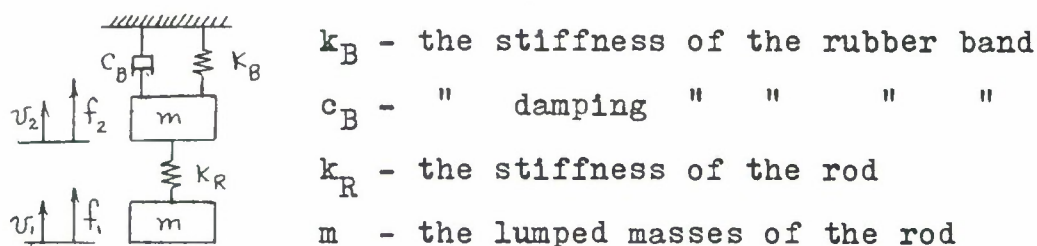
The Lumped Parameter Model of the Rubber Band and Rod Ensemble

Figure 4.3

The impedance matrix, $[Z]$, of the ensemble is written as the sum of a mass matrix, $[M]$, a damping matrix, $[C]$, and a stiffness matrix, $[K]$. The damping in the rod is assumed to be zero; the damping in the rubber band is assumed to be viscous.

$$[Z] = i\omega [M] + [C] + (i\omega)^{-1} [K] \tag{4.1}$$

$$[Z] = i\omega \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C_B \end{bmatrix} + (i\omega)^{-1} \begin{bmatrix} K_R & -K_R \\ -K_R & K_R + k_B \end{bmatrix} \tag{4.2}$$

The admittance matrix for the system is found by inverting the impedance matrix. The elements of the admittance matrix are shown in equations 4.5 through 4.7, and the parameters w_1 and w_2 are defined in equations 4.3 and 4.4.

$$W_1 = \left\{ \frac{K_B}{2m} \right\}^{1/2} \tag{4.3}$$

$$W_2 = \left\{ \frac{2K_R}{m} \right\}^{1/2} \tag{4.4}$$

$$Y_{11}(\omega) = \frac{\frac{\omega}{m}\left[\frac{\omega^2}{W_2^2} - \frac{1}{2}\left(1 + \frac{W_1^2}{W_2^2}\right) \right] + \frac{C_B \omega^2}{m^2 W_2^2}}{i\left(\frac{\omega^4}{W_2^2} - \omega^2 - 2\omega^2 \frac{W_1^2}{W_2^2} - W_1^2 \right) + \left(\frac{\omega^2}{W_2^2} - \frac{1}{2} \right) \frac{\omega}{m} C_B} \tag{4.5}$$

$$Y_{21}(\omega) = \frac{-\frac{1}{2}\frac{\omega}{m}}{i\left(\frac{\omega^4}{W_2^2} - \omega^2 - 2\omega^2 \frac{W_1^2}{W_2^2} - W_1^2 \right) + \left(\frac{\omega^2}{W_2^2} - \frac{1}{2} \right) \frac{\omega}{m} C_B} \tag{4.6}$$

$$Y_{22}(\omega) = \frac{\frac{\omega}{m}\left[\frac{\omega^2}{W_2^2} - \frac{1}{2}\left(1 + \frac{W_1^2}{W_2^2}\right) \right]}{i\left(\frac{\omega^4}{W_2^2} - \omega^2 - 2\omega^2 \frac{W_1^2}{W_2^2} - W_1^2 \right) + \left(\frac{\omega^2}{W_2^2} - \frac{1}{2} \right) \frac{\omega}{m} C_B} \tag{4.7}$$

$$\frac{W_1^2}{W_2^2} = \frac{K_B}{4K_R} \tag{4.8}$$

For comparison the admittances of a free rod, modeled by a spring and two masses, are given in equations 4.9 and 4.10.
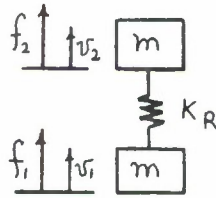
Lumped Parameter Model of a Free Rod

Figure 4.4

$$Y_{11}(\omega) = Y_{22}(\omega) = \frac{\frac{\omega}{m}\left(\frac{\omega^2}{\omega_2^{*2}} - \frac{1}{2}\right)}{i\left(\frac{\omega^4}{\omega_2^{*2}} - \omega^2\right)} \tag{4.9}$$

$$Y_{12}(\omega) = Y_{21}(\omega) = \frac{-\frac{1}{2}\frac{\omega}{m}}{i\left(\frac{\omega^4}{\omega_2^{*2}} - \omega^2\right)} \tag{4.10}$$

ω_2^* is the first resonant frequency of the free rod for elastic motion.

$$\omega_2^* = W_2 = \left\{\frac{2K_R}{m}\right\}^{1/2} \tag{4.11}$$

If $\frac{W_1^2}{W_2^2}$ and C_B are zero, the admittances for the rod and rubber band ensemble are equal to the admittances for the free rod. For the system shown in Figure 4.2, $\frac{W_1^2}{W_2^2}$ is 5.6×10^{-7} and c_B is .403 $\frac{\text{lb.-sec.}}{\text{in.}}$. This value of c_B is 17% of critical damping.

$\frac{W_1^2}{W_2^2}$ is derived using a calculated stiffness for the rod and a measured stiffness for the rubber band.

$$K_R = \frac{AE}{L} = 1.47 \times 10^5 \ \frac{lb.}{in.} \tag{4.12}$$

$$K_B = .33 \ \frac{lb.}{in.} \tag{4.13}$$

$$\frac{W_1^2}{W_2^2} = \frac{K_B}{4K_R} = 5.6 \times 10^{-7} \tag{4.14}$$

c_B is calculated by observing the rate of decay of motion for the rod and rubber band ensemble oscillating in its first mode.

By substituting the values of $\frac{W_1^2}{W_2^2}$ and c_B into equations 4.5 through 4.7 the effect of the rubber band support system on the admittances is determined.

$$Y_{11}(\omega) = \frac{\frac{\omega}{m}\left[\frac{\omega^2}{W_2^2} - \frac{1}{2}\left(1 + 11.2 \times 10^{-7}\right)\right] + \frac{\omega^2}{m^2 W_2^2} .403}{i\left(\frac{\omega^4}{W_2^2} - \omega^2 - \omega^2(11.2 \times 10^{-7}) - W_1^2\right) + \left(\frac{\omega^2}{W_2^2} - \frac{1}{2}\right)\frac{\omega}{m} .403} \tag{4.15}$$

$$Y_{12}(\omega) = Y_{21}(\omega) = \frac{-\frac{1}{2}\frac{\omega}{m}}{i\left(\frac{\omega^4}{W_2^2} - \omega^2 - \omega^2(11.2 \times 10^{-7}) - W_1^2\right) + \left(\frac{\omega^2}{W_2^2} - \frac{1}{2}\right)\frac{\omega}{m} .403} \tag{4.16}$$

$$Y_{22}(\omega) = \frac{\frac{\omega}{m}\left[\frac{\omega^2}{W_2^2} - \frac{1}{2}\left(1 + 11.2 \times 10^{-7}\right)\right]}{i\left(\frac{\omega^4}{W_2^2} - \omega^2 - \omega^2(11.2 \times 10^{-7}) - W_1^2\right) + \left(\frac{\omega^2}{W_2^2} - \frac{1}{2}\right)\frac{\omega}{m} .403} \tag{4.17}$$

If ω, the circular frequency of excitation, is much greater than w_1, the admittances are affected very slightly by the stiffness of the rubber band. At the first resonant

frequency of the free rod, w_2, the magnitudes of the admittances are very sensitive to the values of c_B.

The additional mass of the shaker and impedance head attached to the lower end of the rod, Figure 4.2, also affects the measured admittances. The impedance matrix for a two-degree of freedom, free rod with the mass of the shaker and impedance head, m_s, added is given in equation 4.18.

$$[Z] = (i\omega)\begin{bmatrix} m+m_s & 0. \\ 0. & m \end{bmatrix} + (i\omega)^{-1}\begin{bmatrix} K_R & -K_R \\ -K_R & K_R \end{bmatrix} \tag{4.18}$$

The expression for the first resonant frequency for elastic motion is $\hat{\omega}_1$.

$$\hat{\omega}_1 = \left\{\frac{2K_R}{m}\right\}^{1/2}\left\{\frac{1+\frac{m_s}{2m}}{1+\frac{m_s}{m}}\right\}^{1/2} = \omega_2^*\left\{\frac{1+\frac{m_s}{2m}}{1+\frac{m_s}{m}}\right\}^{1/2} \tag{4.19}$$

If $\frac{m_s}{m} \ll 1$, the resonant frequency of the rod with the shaker and impedance head attached is not appreciably affected by the additional mass, m_s. However, $\frac{m_s}{m}$ for the ensemble in Figure 4.2 is .786, and the first resonant frequency for the 40-inch rod with the shaker and impedance head attached should be significantly lower than ω_2^*.

$$\hat{\omega}_1 = .886\,\omega_2^* \tag{4.20}$$

By modeling the rod with more degrees of freedom it can be

shown that the higher resonant frequencies of the rod will be lower if the shaker and impedance head are attached.

The magnitude and phase of the measured admittances are affected appreciably by the mass of the shaker and impedance head when $\omega = \omega_2^*$.

$$Y_{\shortparallel}(\omega) = \frac{\left(\frac{\omega^2}{\omega_2^{*2}} - \frac{1}{2}\right)}{i\omega m\left[\left(\frac{\omega^2}{\omega_2^{*2}} - 1\right) + \frac{m_s}{m}\left(\frac{\omega^2}{\omega_2^{*2}} - \frac{1}{2}\right)\right]} \tag{4.21}$$

$$Y_{\shortparallel}(\omega_2^*) = (i\omega_2^* m_s)^{-1} \tag{4.22}$$

By comparing the measured admittances for the total ensemble of the rod, rubber band, shaker and impedance head to admittances calculated for a 40-inch free rod, the effects of the suspension system, shaker and impedance head on the measured admittances are evident. See Figures 4.6 through 4.9.

The calculated admittances are generated by solving equation 4.23 for a free rod.

$$E \frac{\partial^2 u}{\partial x^2} + \rho \frac{\partial^2 u}{\partial t^2} = 0. \tag{4.23}$$

$$\frac{\partial u(0)}{\partial x} = 0. \tag{4.24}$$

$$\frac{\partial u(L)}{\partial x} = 0. \tag{4.25}$$
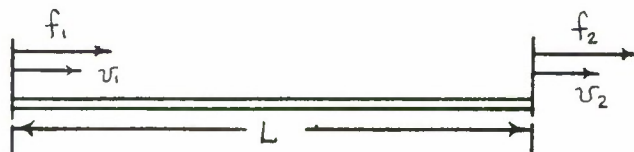
Schematic of a Free Rod

Figure 4.5
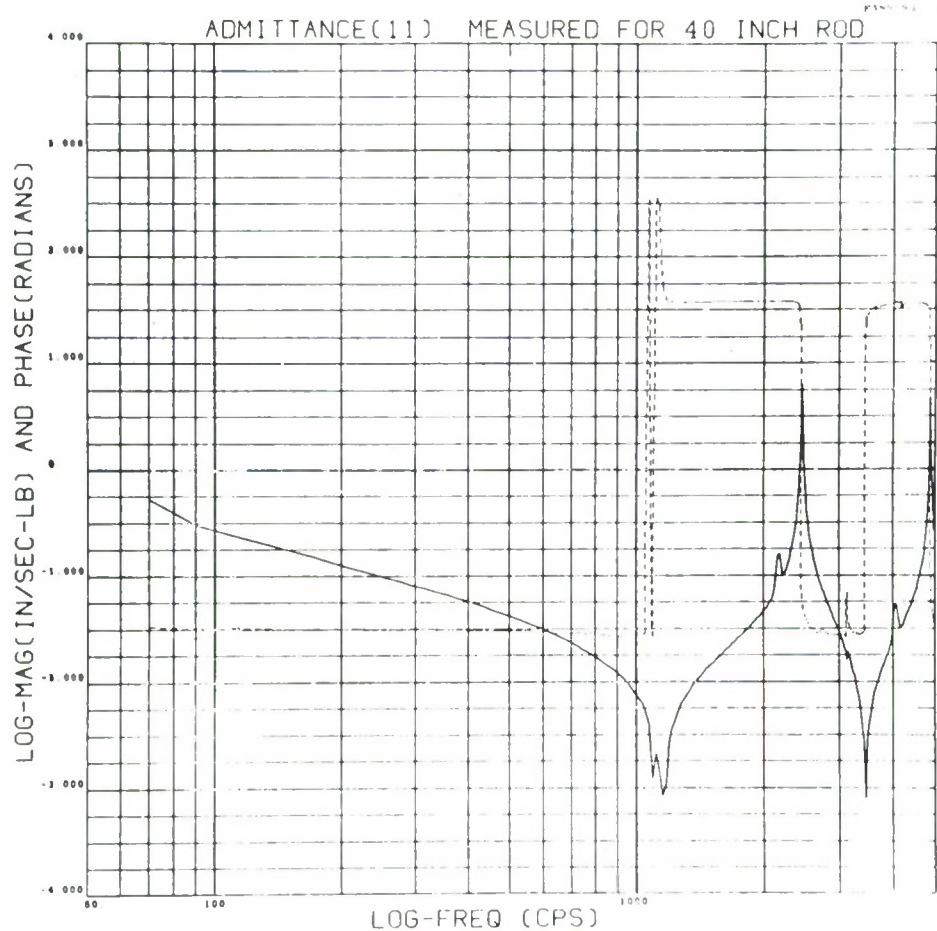
Figure 4.6

Note: On all admittance plots of the type shown above,
the broken line is phase and the solid line is magnitude.
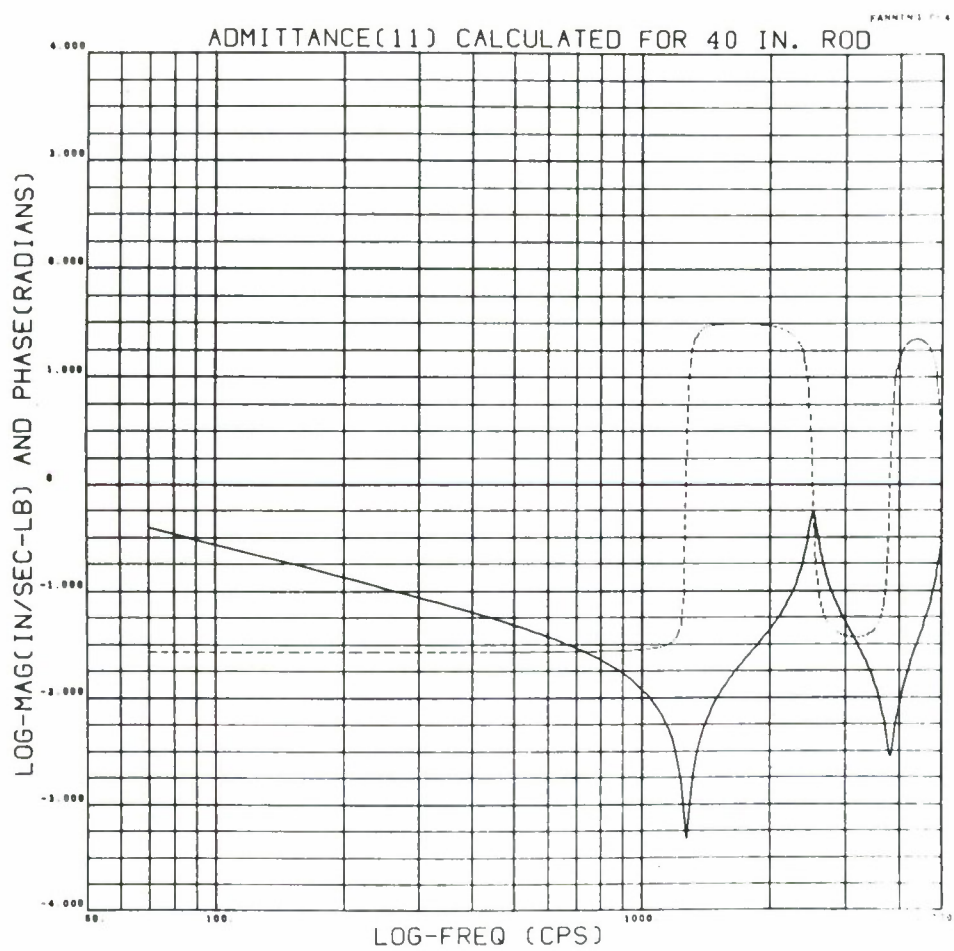
Figure 4.7
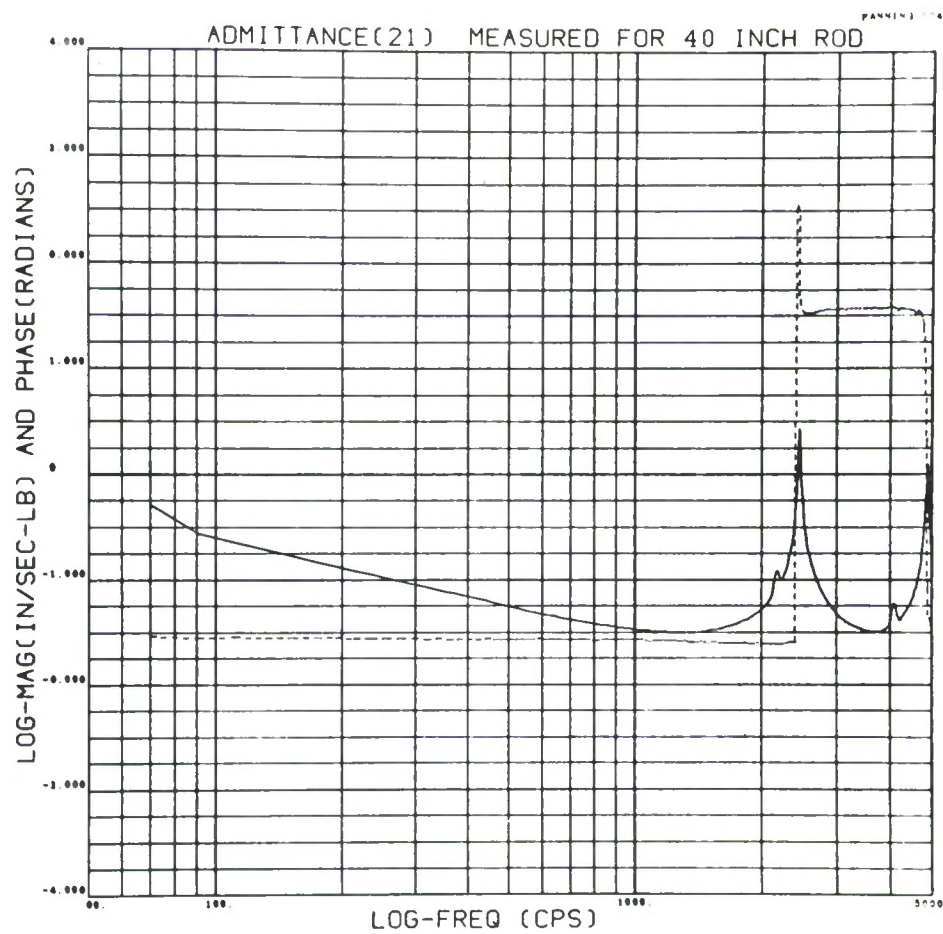
Figure 4.8

58
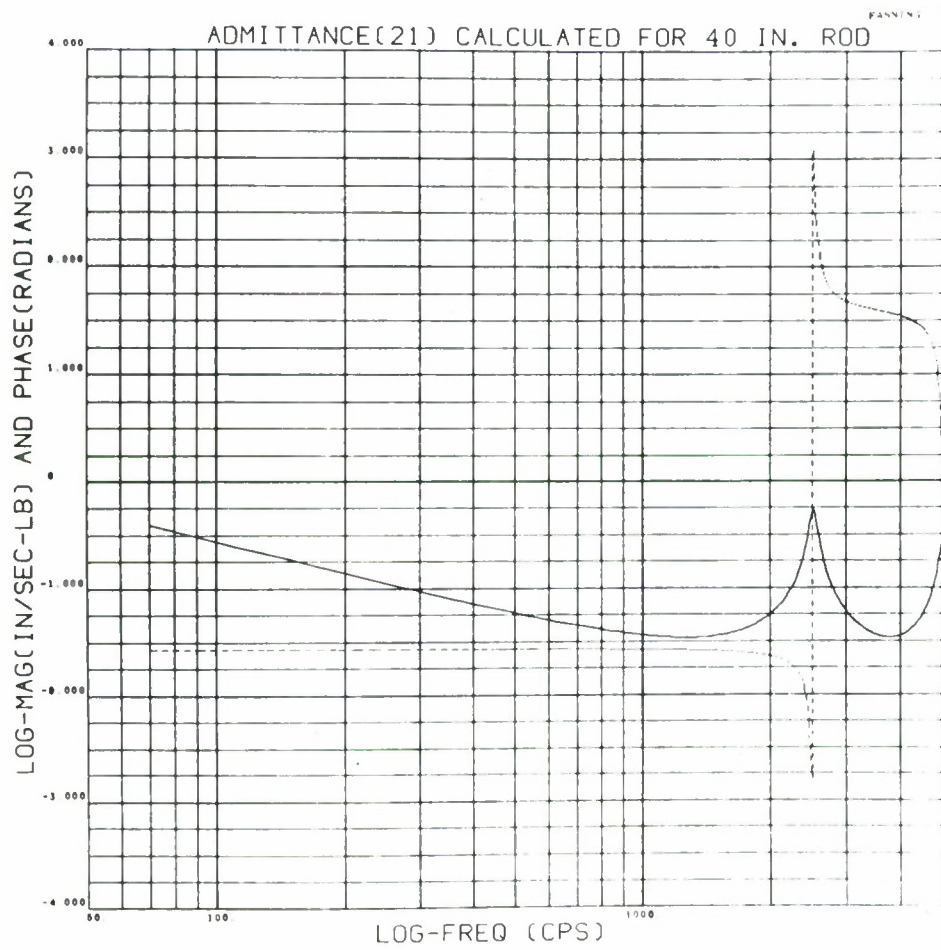
Figure 4.9

u is the axial displacement of the rod and is a function of the location, x, and time, t.

$$E \text{ - Young's Modulus} \qquad 29.8 \times 10^6 \text{ psi}$$
$$A \text{ - Cross Sectional Area} \qquad .198 \text{ in.}^2$$
$$\rho \text{ - Density} \qquad .284 \frac{\text{lb.}}{\text{in.}^3}$$
$$L \text{ - Length} \qquad 40. \text{ in.}$$

Properties of a 40-Inch

Steel Rod

Table 4.1

The solutions to equation 4.23 incorporating the boundary conditions in equations 4.24 and 4.25 are given in equation 4.26.

$$\phi^n(x) = \cos \frac{n\pi x}{L} \tag{4.26}$$

The admittances of the free rod are constructed in a manner similar to the method used in section 2.1.

$$Y_{m\ell}(\omega) = \sum_{n=0}^{\infty} \frac{\frac{\omega}{m_n} \phi^n(x_m) \phi^n(x_\ell)}{\zeta_n(\omega)\omega + i(\omega^2 - \omega_n^2)} \tag{4.27}$$

$$m_n = \int_0^L \rho A \cos^2 \frac{n\pi x}{L} \, dx \tag{4.28}$$

$$\omega_n = \frac{n\pi}{L}\left(\frac{E}{\rho}\right)^{1/2} \tag{4.29}$$

$$\zeta_n(\omega) = .02\,\omega_n \tag{4.30}$$

The admittances of the free rod are calculated for 242 frequencies from 50 c.p.s. to 5000 c.p.s. These are the same frequencies at which the measured admittances are sampled and scaled. The damping is equivalent to 1% of critical damping.

4.3 Predicting the Admittances, Resonant Frequencies and Mode Shapes for an 80-Inch Free Rod

The mutual and cross admittances for a 40-inch free rod are measured as shown in Figure 4.10. The mutual admittance is measured using the accelerometer in the impedance head. The cross admittance is measured using an accelerometer attached to the top end of the rod.
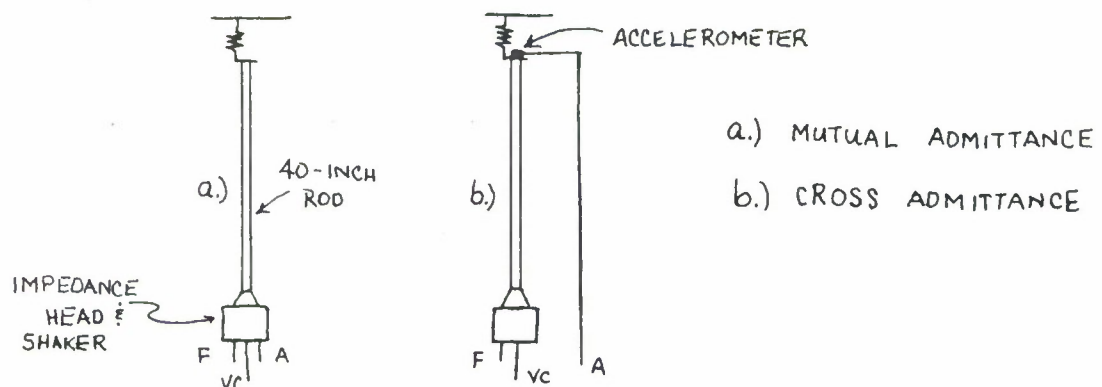
Schematic of Measuring the Admittances for a 40-Inch Rod

Figure 4.10

Having measured the mutual and cross admittances is sufficient to construct two identical, 2×2, substructure admittance matrices. The mutual admittance of one end of the rod is the same as the mutual admittance of the other end; the cross admittances are the same due to symmetry in the admittance matrix. Assuming that admittance measurements made on one 40-inch rod should not, in principle, differ from admittance measurements made on a second 40-inch rod, two identical substructure admittance matrices are constructed from admittance measurements made on one 40-inch rod.

The two admittance matrices for the 40-inch rod sub-
structures are inverted to substructure impedance matrices.
The substructure impedance matrices are assembled, as shown
in section 2.2, to yield the impedance matrix of the compos-
ite structure, an 80-inch free rod.
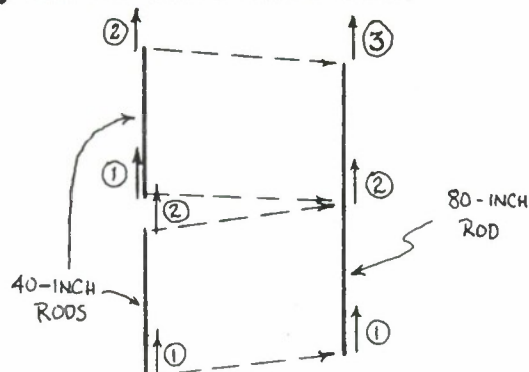
Schematic of Assembling an 80-Inch Rod from Two 40-Inch Rods

Figure 4.11

The impedance matrix for the 80-inch rod is inverted to
the composite structure admittance matrix. The four unique
elements of the composite structure admittance matrix are
plotted as a function of the frequency of excitation in
Figures 4.12 through 4.15. Figures 4.16 through 4.19 are
calculated admittances for an 80-inch free rod. These admit-
tances are calculated using equation 4.27. The damping in
the calculated admittances is 1% of critical damping.

The admittance matrix of the composite structure is
used to estimate resonant frequencies and mode shapes of the
composite structure as shown in section 2.4. The predicted
mode shapes for the 80-inch rod that have corresponding
resonant frequencies that coincide with resonant frequencies
of the substructures are not accurate. See Table 4.2.

Figure 4.12

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Figure 4.13

ADMITTANCE(31) PREDICTED FOR 80 INCH ROD

LOG-FREQ (CPS)

LOG-MAG(IN/SEC-LB) AND PHASE(RADIANS)
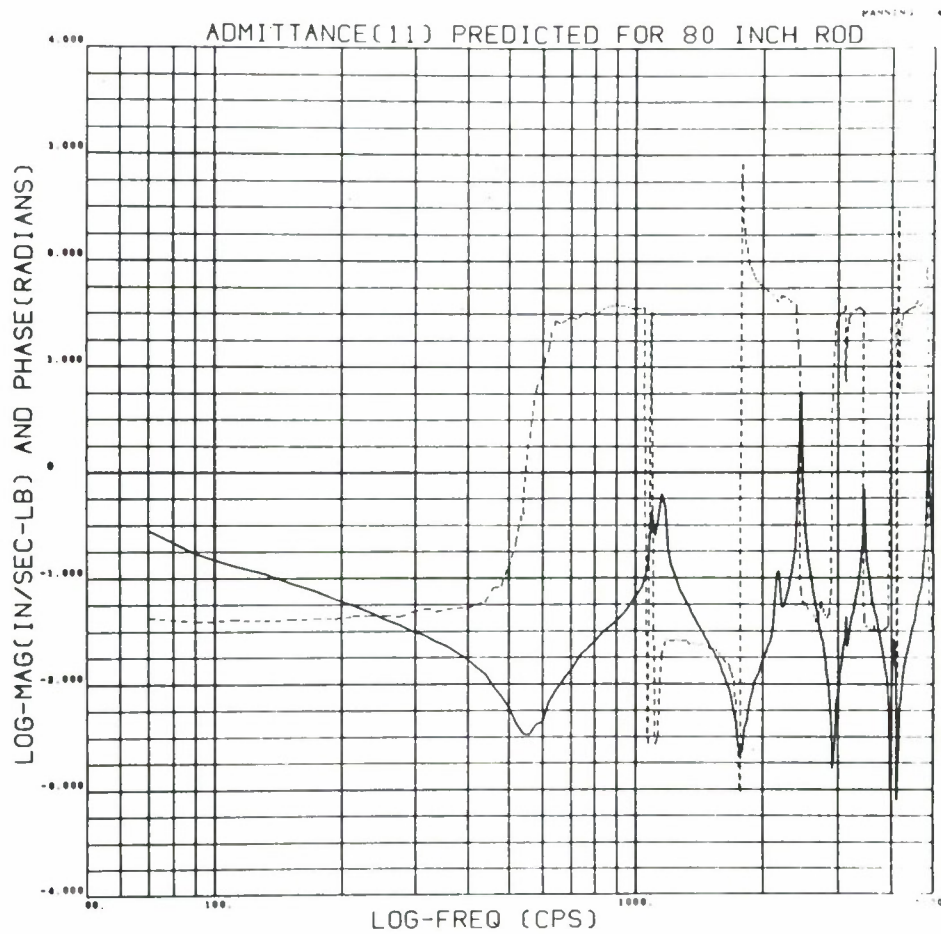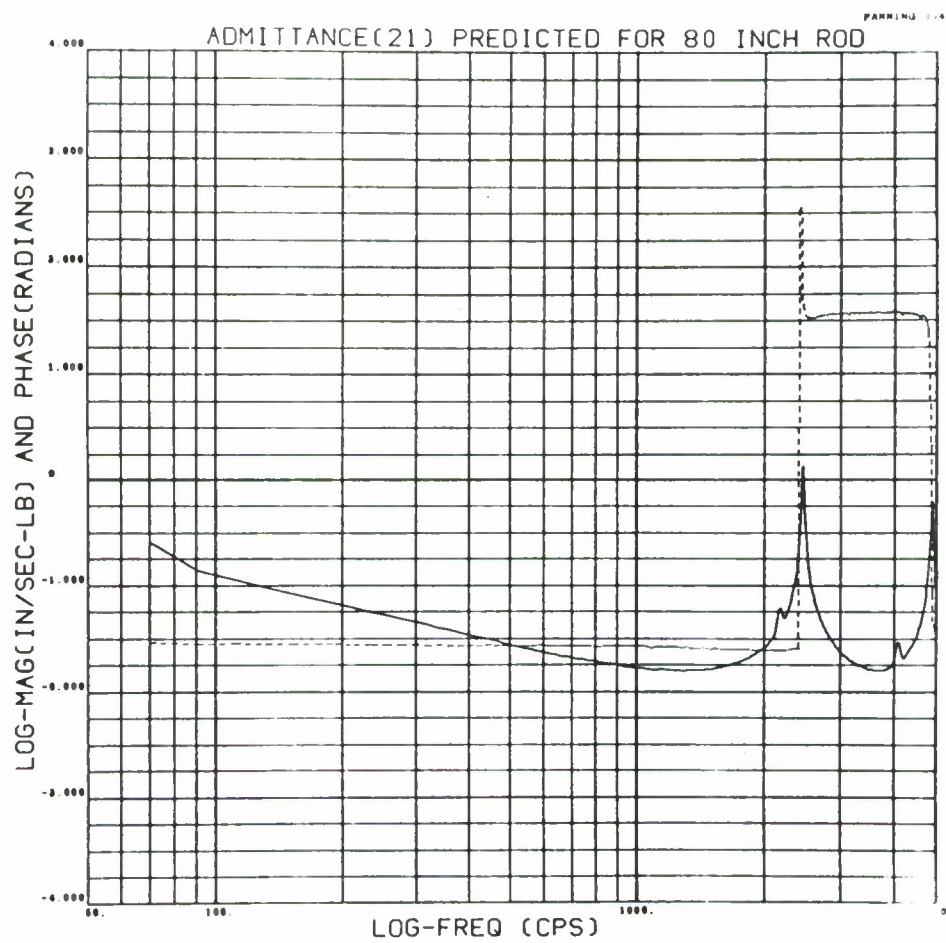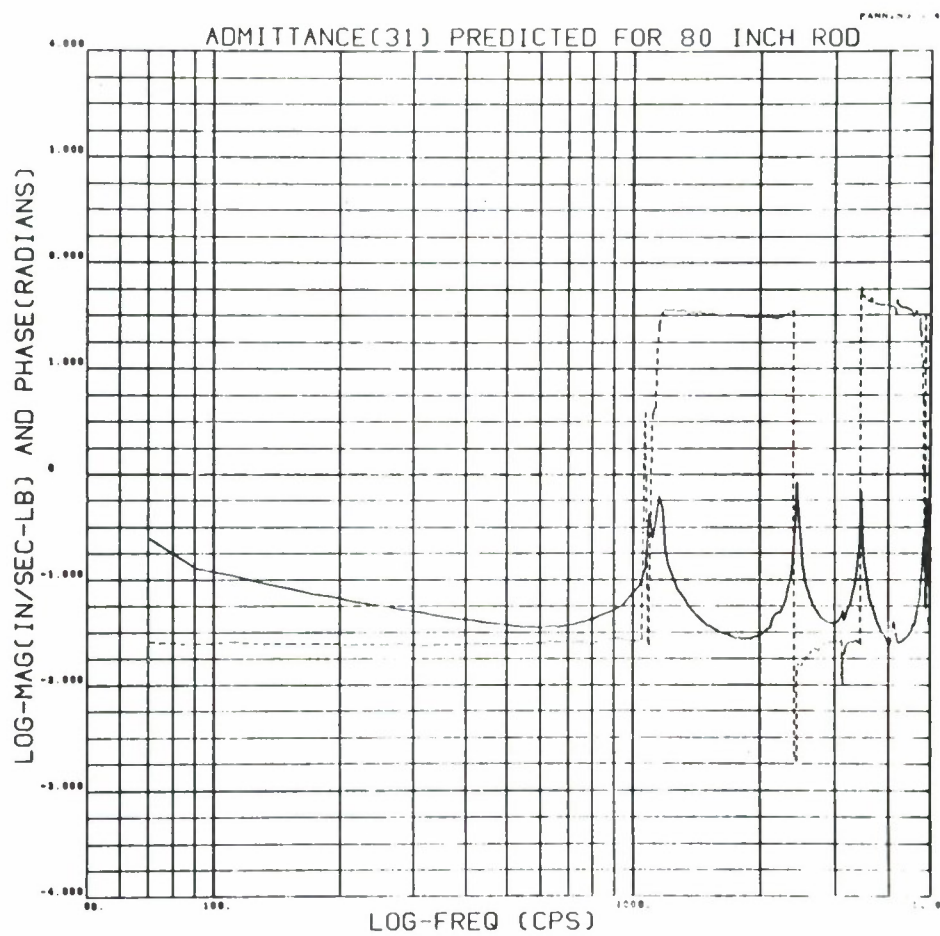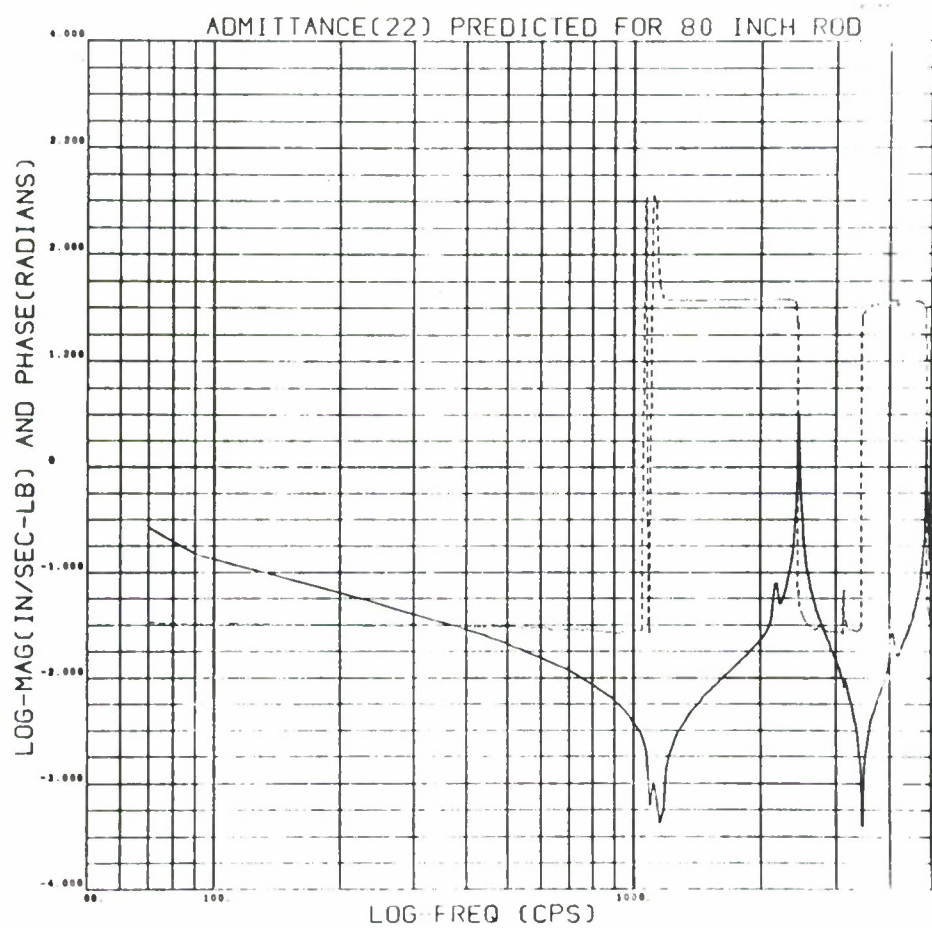
Figure 4.14

Figure 4.15

Figure 4.16

68
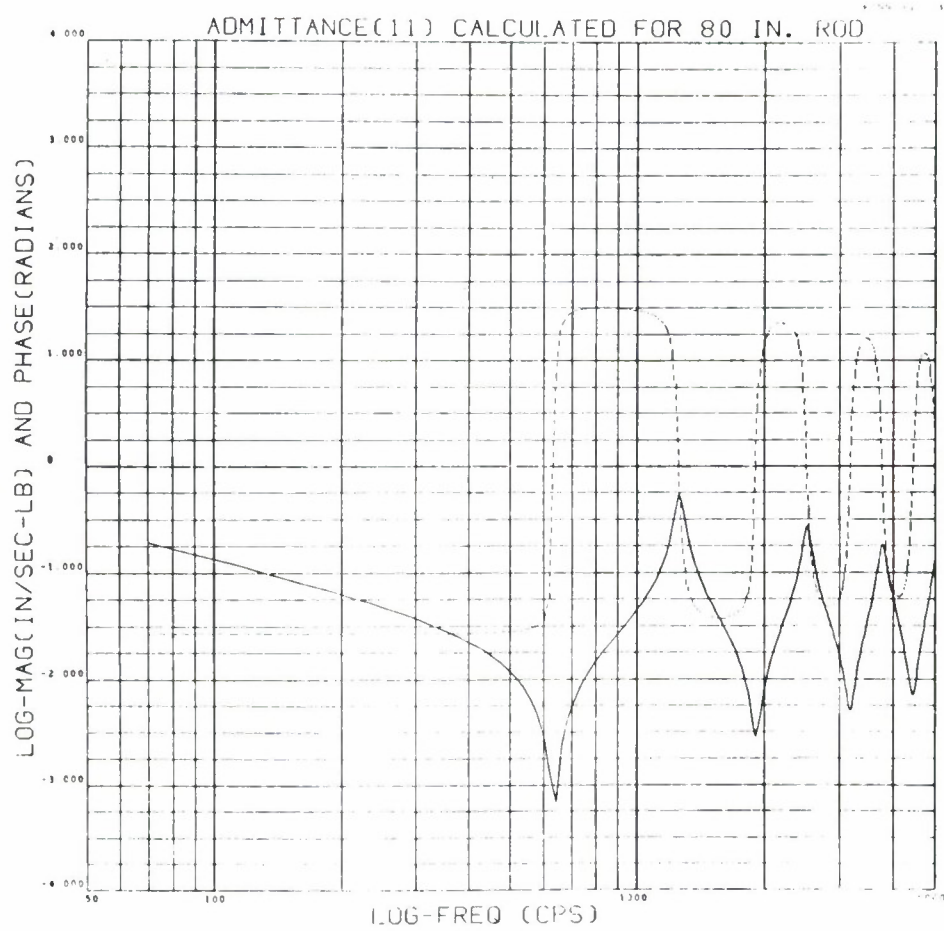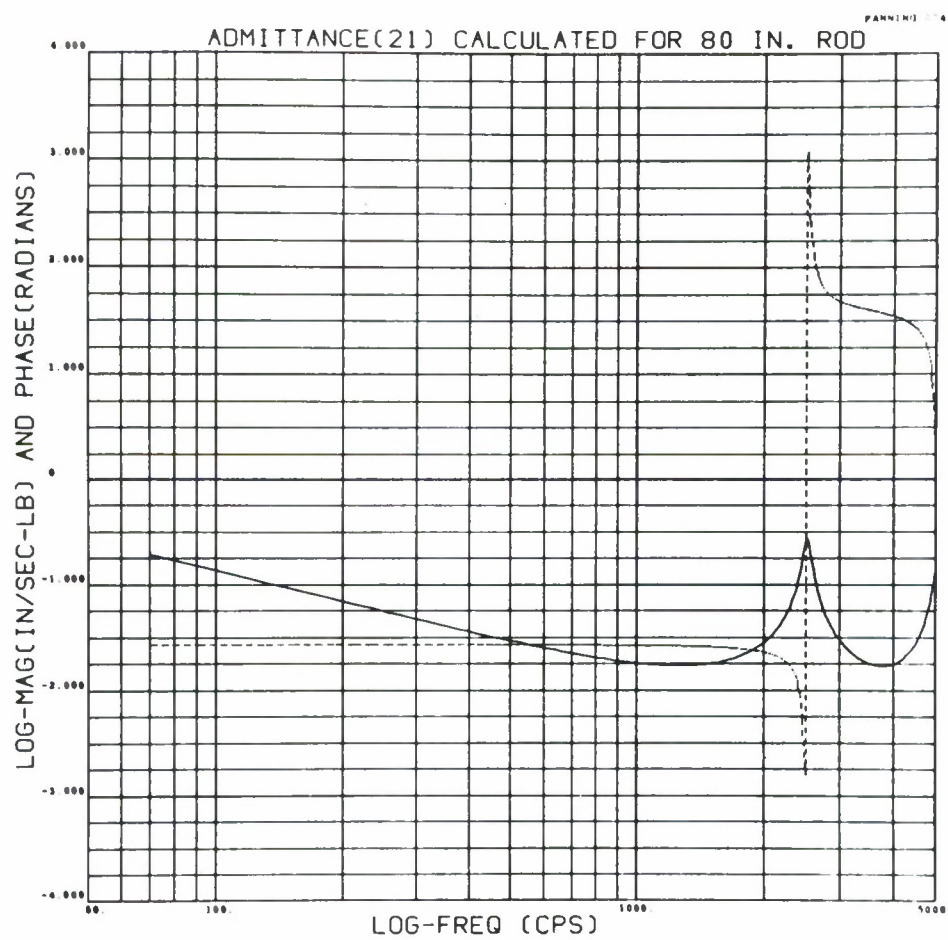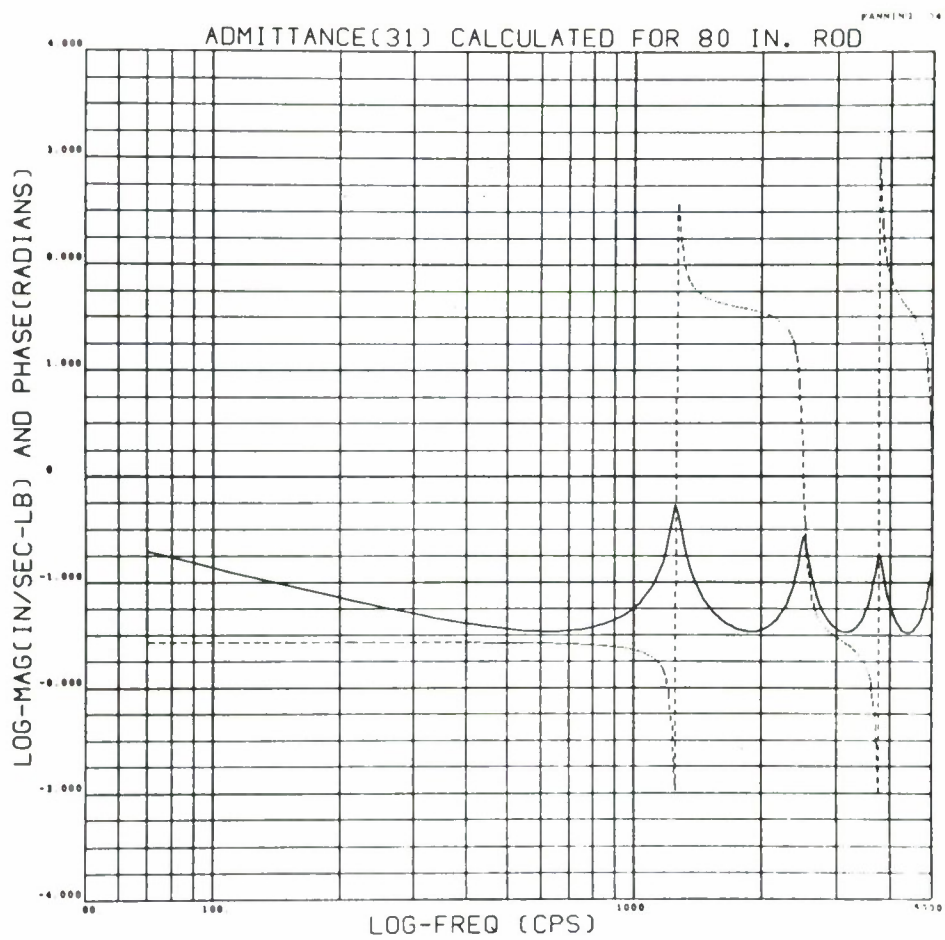
Figure 4.17

Figure 4.18
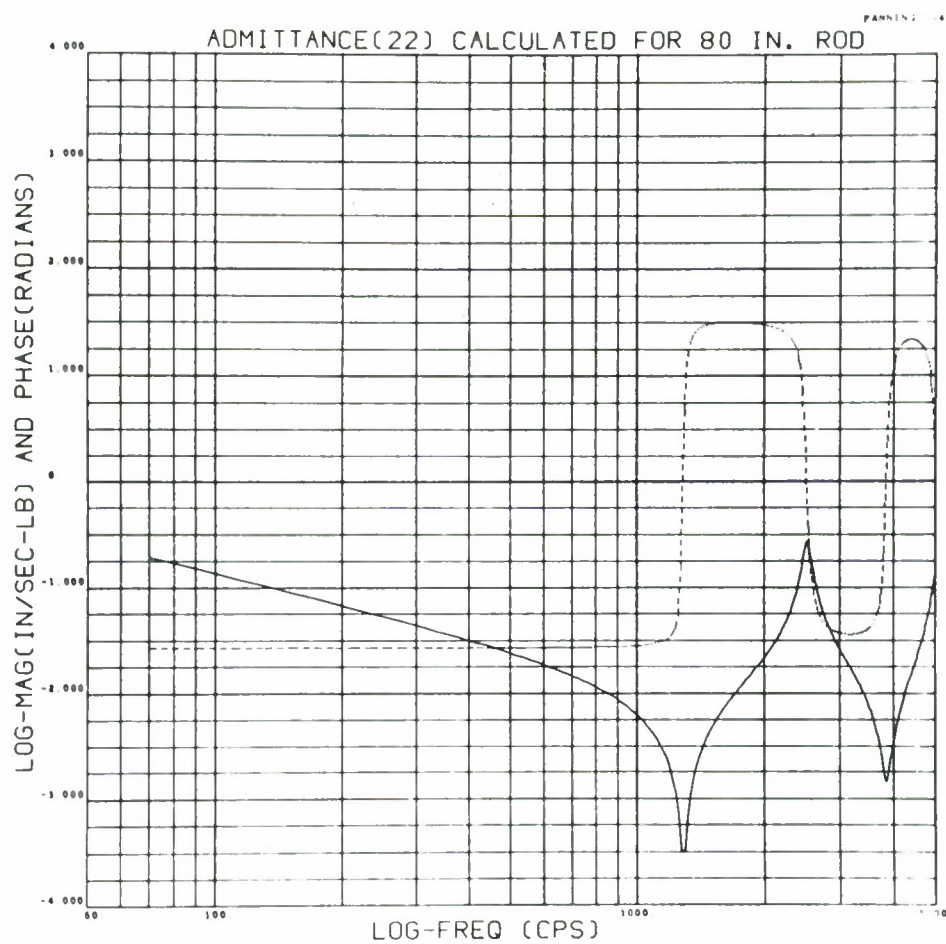
Figure 4.19

Calculated and Predicted Mode Shapes

and Resonant Frequencies

Table 4.2

	Location	1	2	3	Resonant Frequency (c.p.s.)
	Admittance	11	21	31	
Mode Shape					
1	Predicted	1.0	.053*	- .998	1134.
	Calculated	1.0	0.0	- 1.0	1257.
2	Predicted	1.0	-.238	.142	2422.**
	Calculated	1.0	- 1.0	1.0	2515.
3	Predicted	.996	.025*	- 1.0	3527.
	Calculated	1.0	0.0	- 1.0	3772.
4	Predicted	1.0	-.012	.012	4815.**
	Calculated	1.0	- 1.0	1.0	5030.

* No local maxima appear in this element of the predicted admittance matrix at or near this frequency. See section 2.4.

** These frequencies are very close to resonant frequencies of the substructures.

Equations 4.15 through 4.17 show that the measured admittances of the 40-inch rod substructure are affected by the damping of the suspension system at the first resonant frequency of the free rod. In addition equation 4.22 shows that the mass of the shaker and impedance head also distorts the admittances at the frequency ω_2^*. Consequently, the admittances predicted from admittance measurements made near the resonant frequencies of the substructures are not accurate. Therefore, if a resonant frequency of a substructure is close to a resonant frequency of the composite structure, one should not expect to predict accurately the corresponding mode shape for the composite structure.

In Table 4.2 the calculated mode shapes, $\phi^n(x)$, are defined in equation 4.26 with L equal to 80 inches. The calculated resonant frequencies are defined as follows:

$$\nu_n = \frac{\omega_n}{2\pi} = \frac{n}{2L}\left(\frac{E}{\rho}\right)^{1/2} \tag{4.31}$$

The predicted mode shapes and resonant frequencies are calculated as shown in section 2.4.

4.4 The Effects of the Suspension System on the Measured Admittances of a 32-Inch Free Beam

The Beam is suspended on three lengths of $\frac{1}{4}$-inch shock cord to simulate a free beam. The shock cord hangs parallel to the Y axis in Figure 4.33. See also Figures 4.20 and 4.21. The beam is suspended on the shock cord such that the majority of the weight of the beam is balanced by the reaction force in the length of cord attached to the beam nearest to the point of forced excitation. Consequently, the suspension is changed every time the shaker and impedance head are attached at a new location.

Figure 4.23 is an analog plot of the mutual admittance of location 2 on the 32-inch beam using a suspension system similar to the one shown in Figure 4.20. Figure 4.24 is an analog plot of the same admittance measured using a suspension system that supports the majority of the weight of the beam at locations not near the point of forced excitation.

Two lightweight bolts, 1.5 inches long, are put into the beam some distance away from and on both sides of the shaker and impedance head attachment point. One length of shock cord is attached to the upper flange of the beam just above the impedance head and shaker, and the other two lengths of cord are attached to the ends of the bolts. Figure 4.25 shows an X,Y plane projection of the suspension system with the shaker and impedance head attached to the beam.
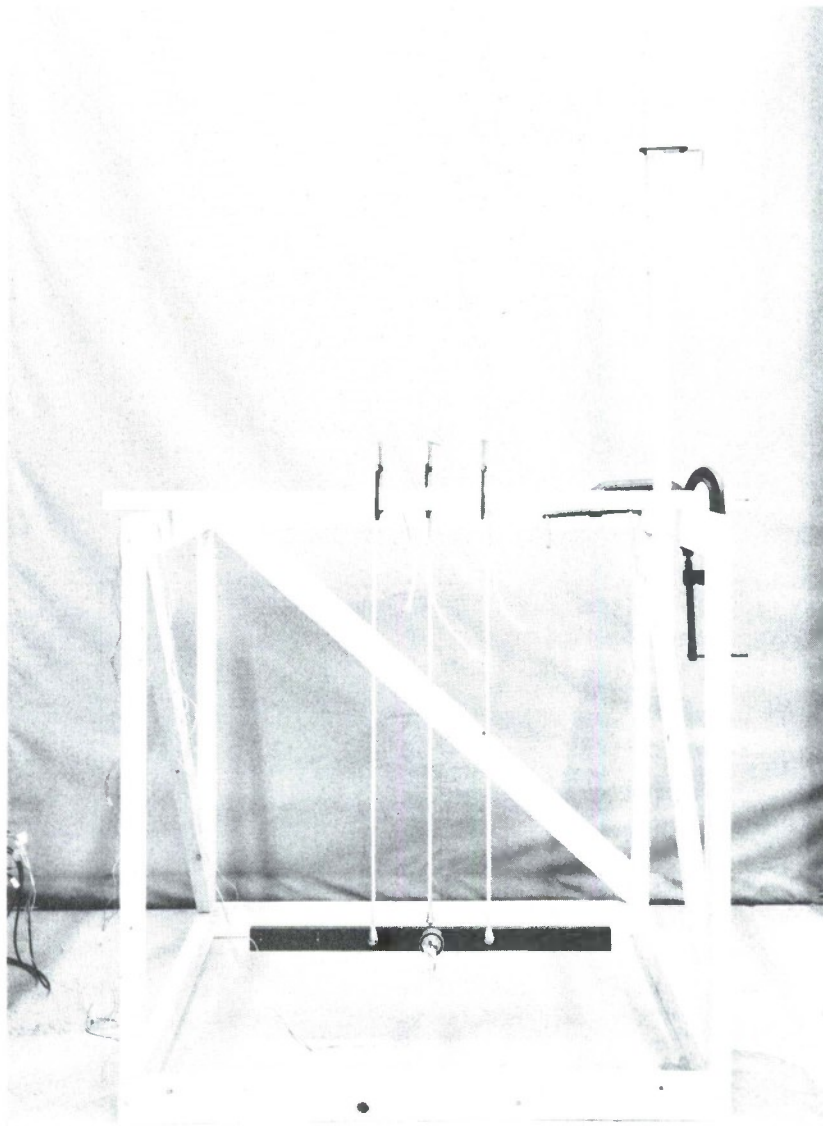
Suspension System Used to Measure the
Admittances of a 32-Inch Free Beam

Figure 4.20
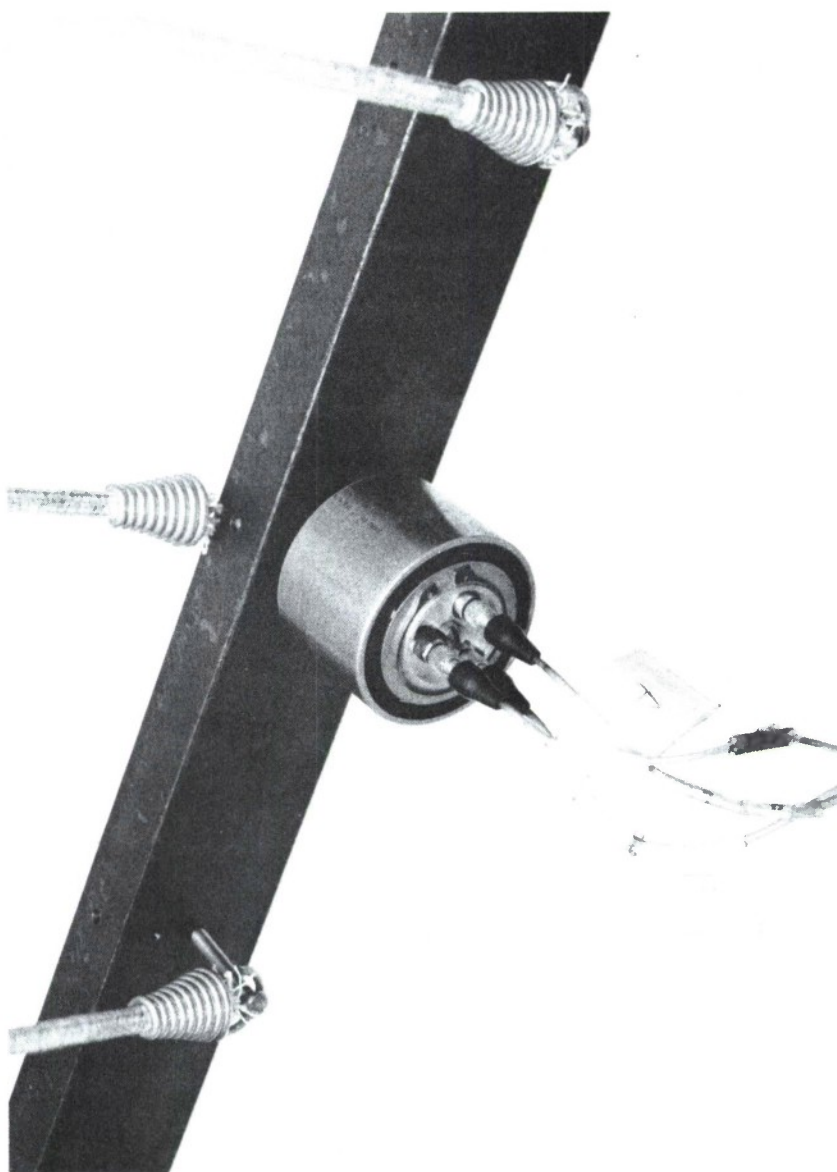
Measuring a Mutual Admittance
of a 32-Inch Free Beam

Figure 4.21
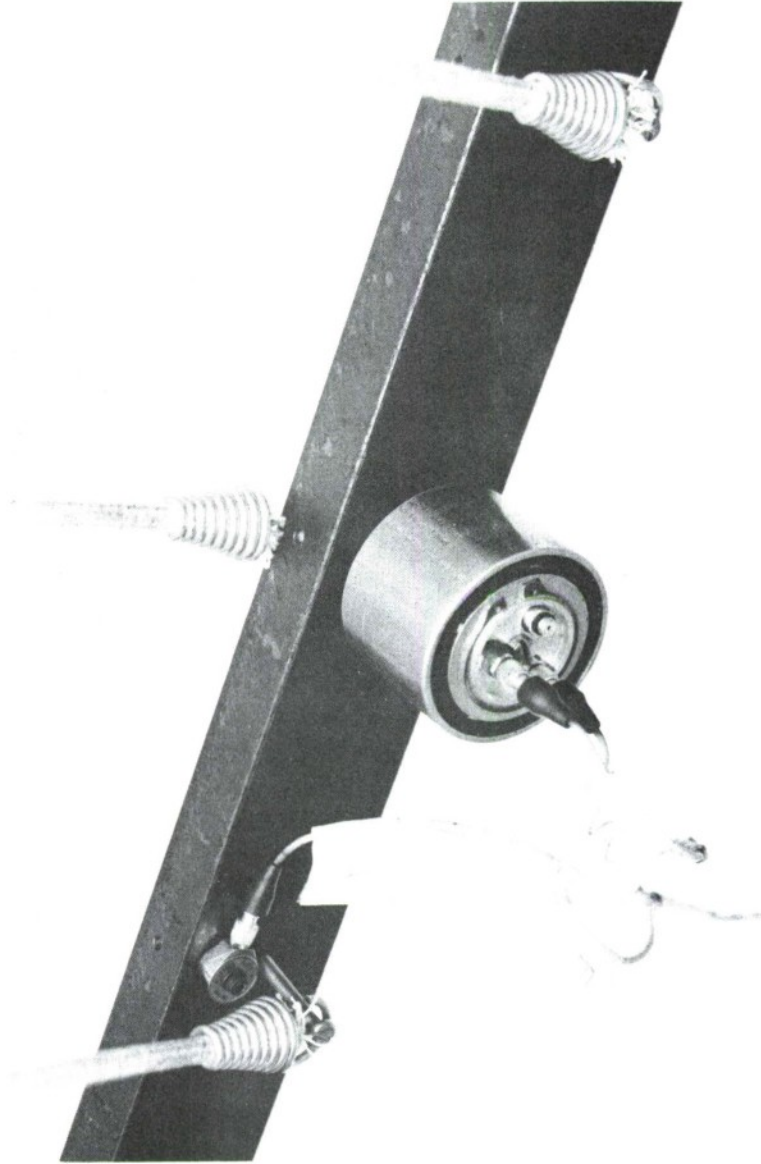
Measuring a Cross Admittance
of a 32-Inch Free Beam

Figure 4.22

SPECTRAL DYNAMICS CORPORATION OF SAN DIEGO

ADMITTANCE (2.2) OF
A 32 INCH BEAM

Figure 4.23

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Figure 4.24

ADMITTANCE (22) OF A 32-INCH BEAM

SPECTRAL DYNAMICS CORPORATION OF SAN DIEGO
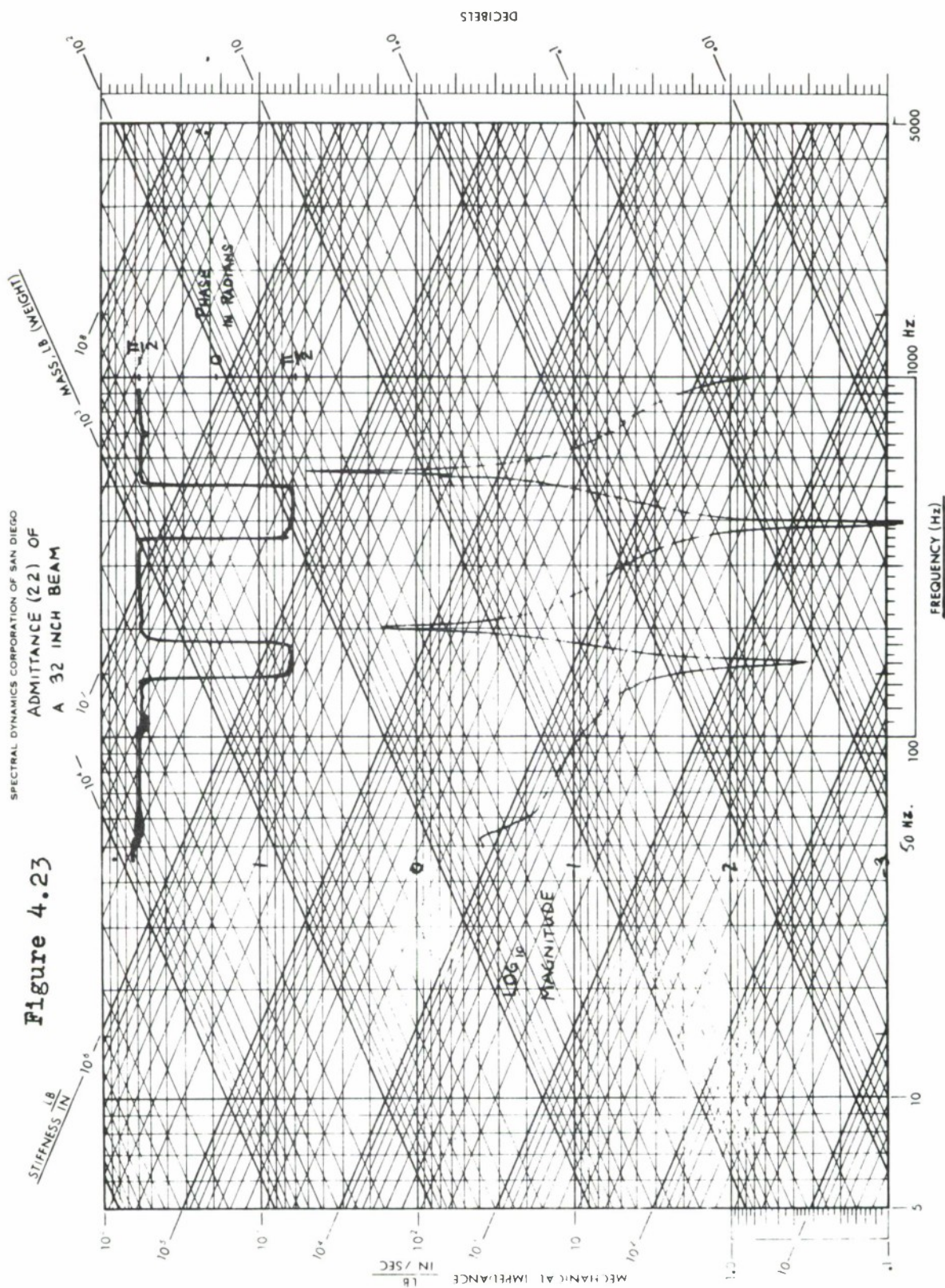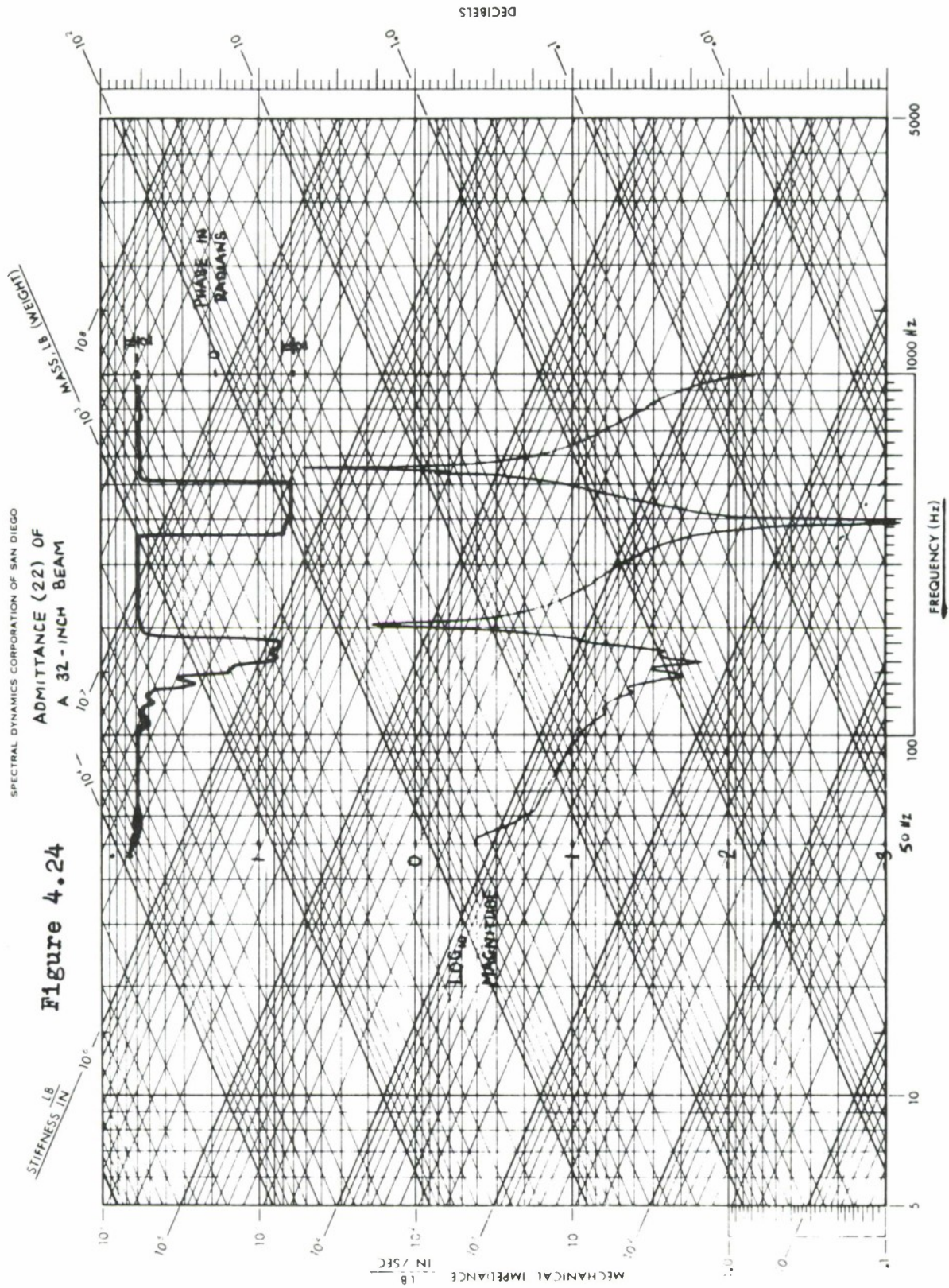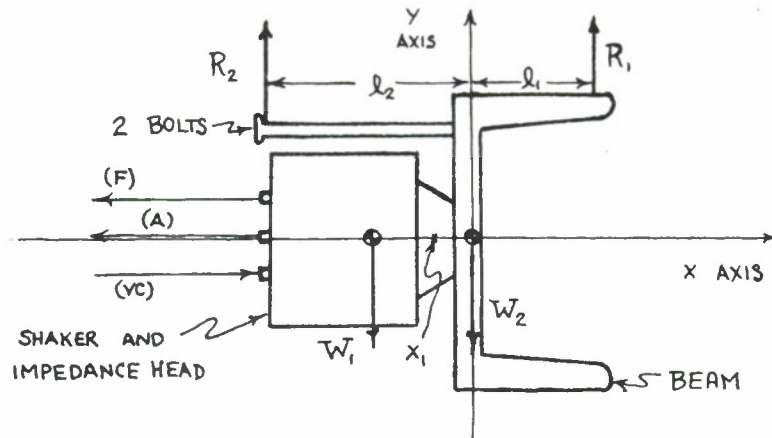
Schematic of 32-Inch Beam Suspension System

Figure 4.25

R_1 is the deadweight reaction force in the shock cord attached to the beam near the point of forced excitation. R_2 is the sum of the deadweight reaction forces in the shock cords attached to the two bolts. W_1 and W_2 are the weights of the shaker with impedance head and the beam, respectively. The location of the center of gravity for the beam, shaker and impedance head is x_1.

Using force equilibrium in the Y direction, equation 4.32, and moment equilibrium in the Z direction, equation 4.33, the deadweight reaction forces, R_1 and R_2, are determined.

$$(R_2 + R_1 - W_1 - W_2) = 0. \tag{4.32}$$

$$-R_2(\ell_2 + x_1) + R_1(\ell_1 - x_1) = 0. \tag{4.33}$$

$$R_1 = (W_1 + W_2)\frac{(\ell_2 + x_1)}{(\ell_1 + \ell_2)} \tag{4.34}$$

$$R_2 = (W_1 + W_2) \frac{(\ell_1 - x_1)}{(\ell_1 + \ell_2)} \qquad\qquad (4.35)$$

Because R_2 is the sum of the two deadweight reaotion forces in the shock cords attached to the bolts and each of the two deadweight reaction forces at the bolts must be positive for the suspension system to be statically stable, R_2 is the largest possible force in either of the two shock cords attached to the bolts. In general, the deadweight reaction forces in the cords attached to the bolts are different.

During the forced excitation of the beam the force in the shock cord changes very little, assuming that the motion of the shock cord is small. Assuming the angle between the shock cord and the vertioal, Θ , is a function of the displacement of the beam where the cord is attached, one can solve for the force exerted in the X direction by the cord on the beam, F_R.
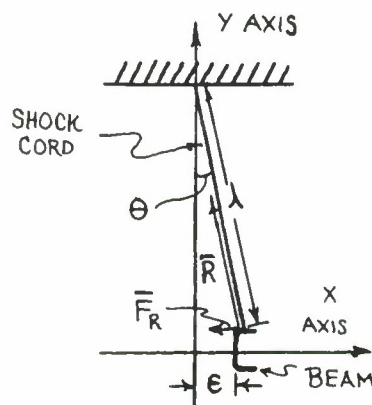
Schematic of the Reaction Forces in the Suspension System

Figure 4.26

$$F_R = -\frac{R\epsilon}{\lambda} \qquad (4.36)$$

F_R is proportional to the deadweight reaction force, R, and the local displacement of the beam, ϵ .

To simulate a free beam it is best to make the deadweight reaction forces as small as possible. The suspension system shown in Figure 4.21 is an attempt to minimize R_2 and make $R_1 \approx W_1 + W_2$. Therefore, the reaction forces, F_R, are small at locations away from the point of excitation and the reaction force at the point of excitation is, hopefully, much smaller than the force applied by the shaker.

If F_S is the force applied to the beam in the X direction by the shaker and F_{RS} is the reaction force in the X direction exerted on the beam by the shock cord attached at the point of excitation, one can solve for the measured mutual admittance.

$$Y(\omega)_{measured} = \left\{ \frac{F_S - F_{RS}}{i\omega\epsilon_S} \right\}^{-1} \qquad (4.37)$$

$$F_{RS} = -\frac{R_S \epsilon_S}{\lambda} \qquad (4.38)$$

ϵ_S is the local displacement of the beam at the point of excitation and R_S is the deadweight reaction force in the shock cord attached to the beam at the point of excitation.

If $R_S = W_1 + W_2$, a bound for the effect of the shock cord on the measured mutual admittance is established.

$$Y(\omega)_{measured} = \left\{ \frac{F_S}{i\omega\epsilon_S} - \frac{F_{RS}}{i\omega\epsilon_S} \right\}^{-1} = \left\{ Y(\omega)_{true}^{-1} + \frac{W_1 + W_2}{\lambda\omega i} \right\}^{-1} = \left\{ Y(\omega)_{true}^{-1} + \frac{.226}{i\omega} \right\}^{-1} \quad (4.39)$$

For higher frequencies of excitation the effect of the dead-weight reaction force, R_S, is very small except at resonant frequencies of the beam where $Y(\omega)_{true}^{-1}$ is also very small. Consequently, the suspension system affects the admittance measurements more at or near the lower resonant frequencies of the 32-inch beam.*

The frequencies of local maxima in the measured admittances that have an accompanying phase shift of π radians are compared to calculated resonant frequencies for the 32-inch free beam.

Resonant Frequency	Calculated[9]	Measured
1	224. c.p.s.	206. c.p.s.
2	628. c.p.s.	549. c.p.s.

Comparison of Measured and Calculated Resonant

Frequencies for a 32-Inch Free Beam

Table 4.3

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* See reference 8 for the calculated admittances of a free beam.
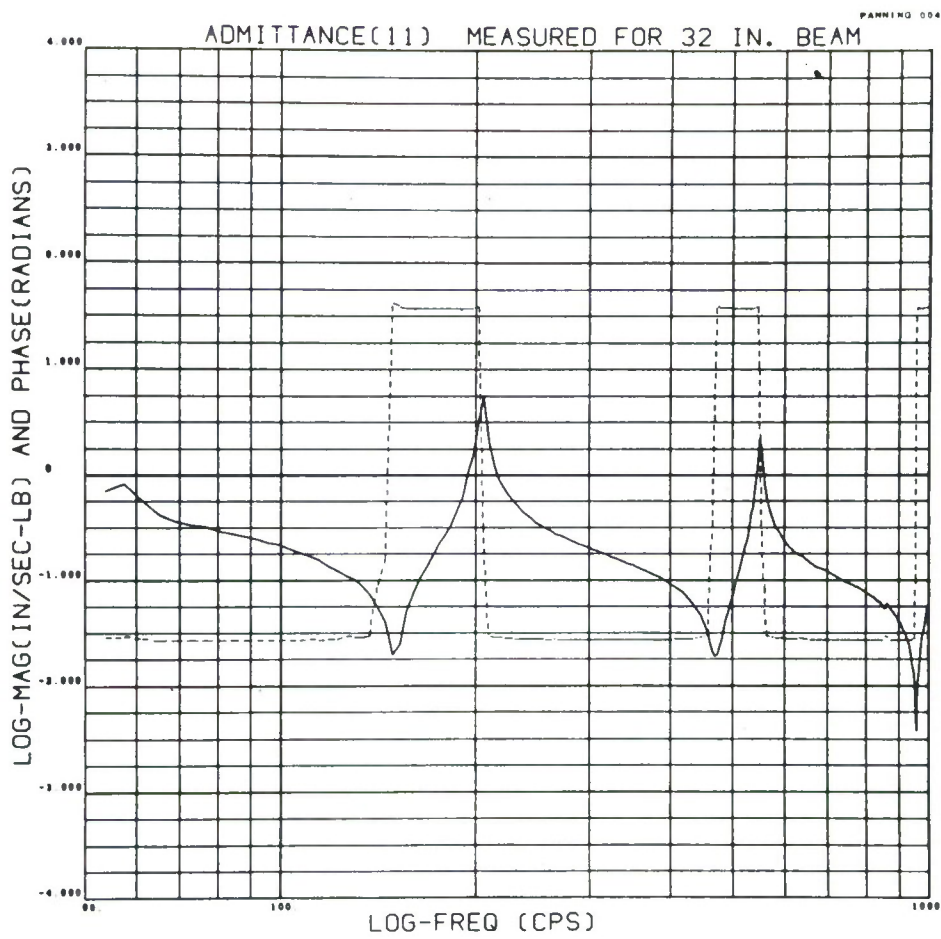
Figure 4.27
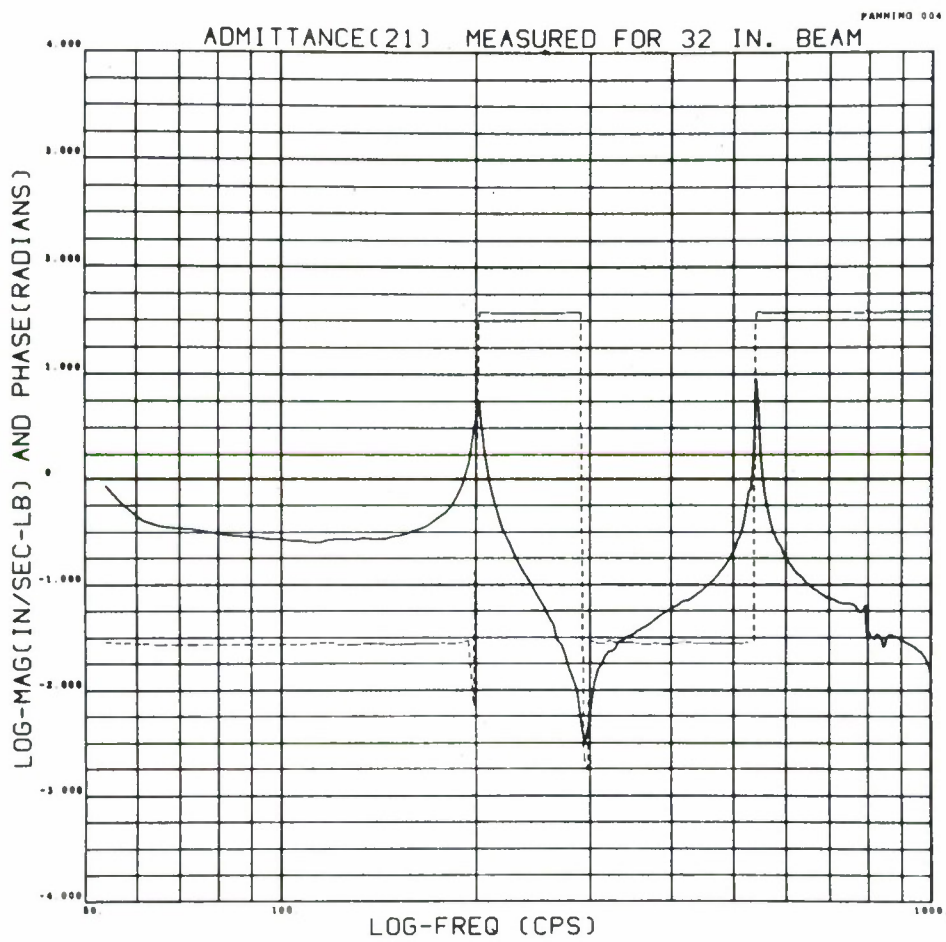
Figure 4.28
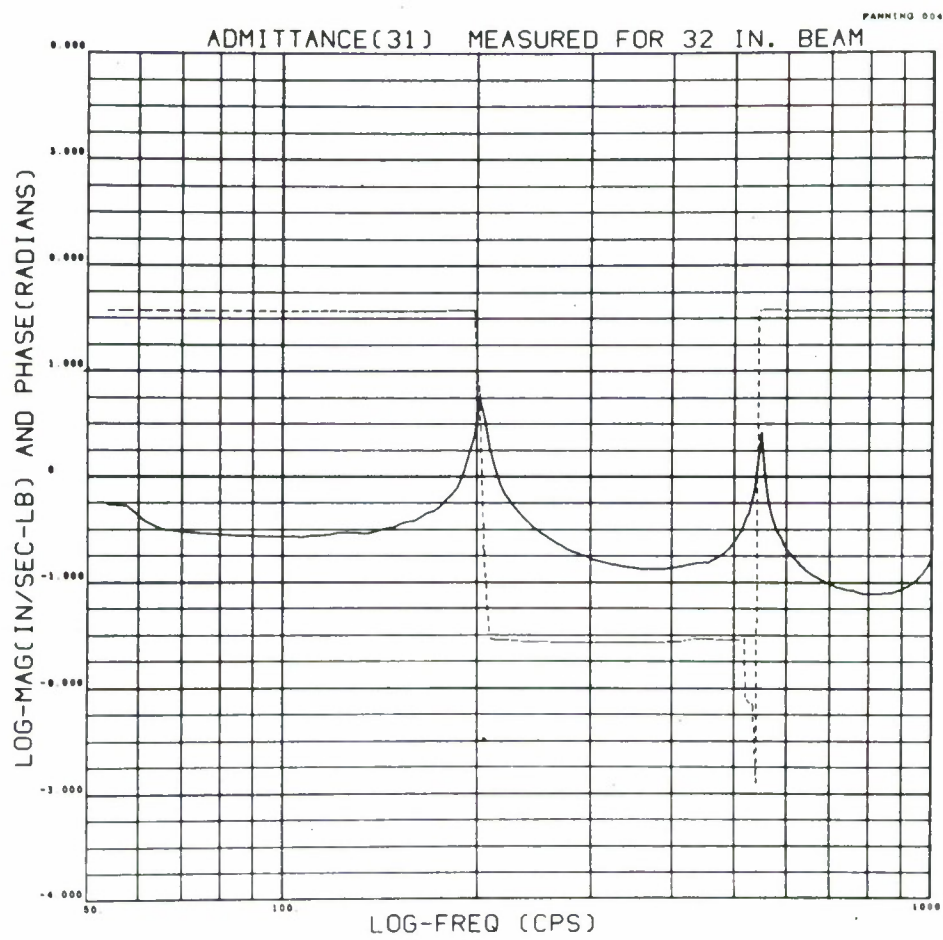
Figure 4.29
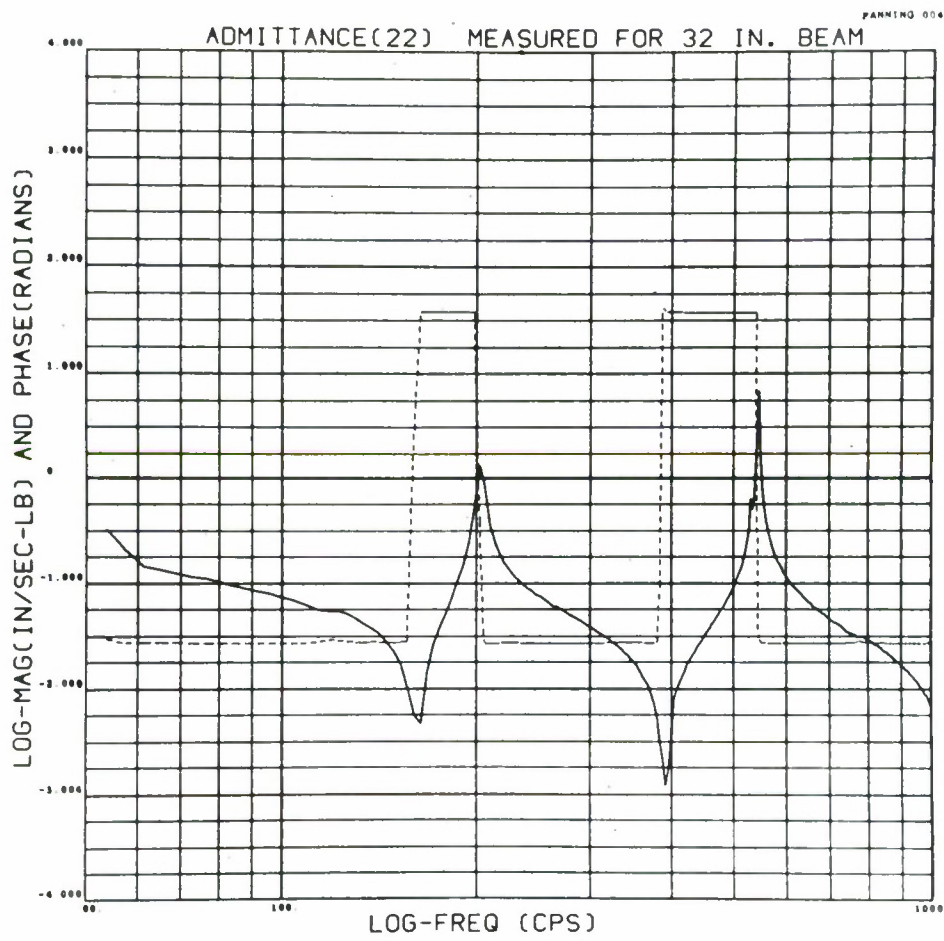
Figure 4.30
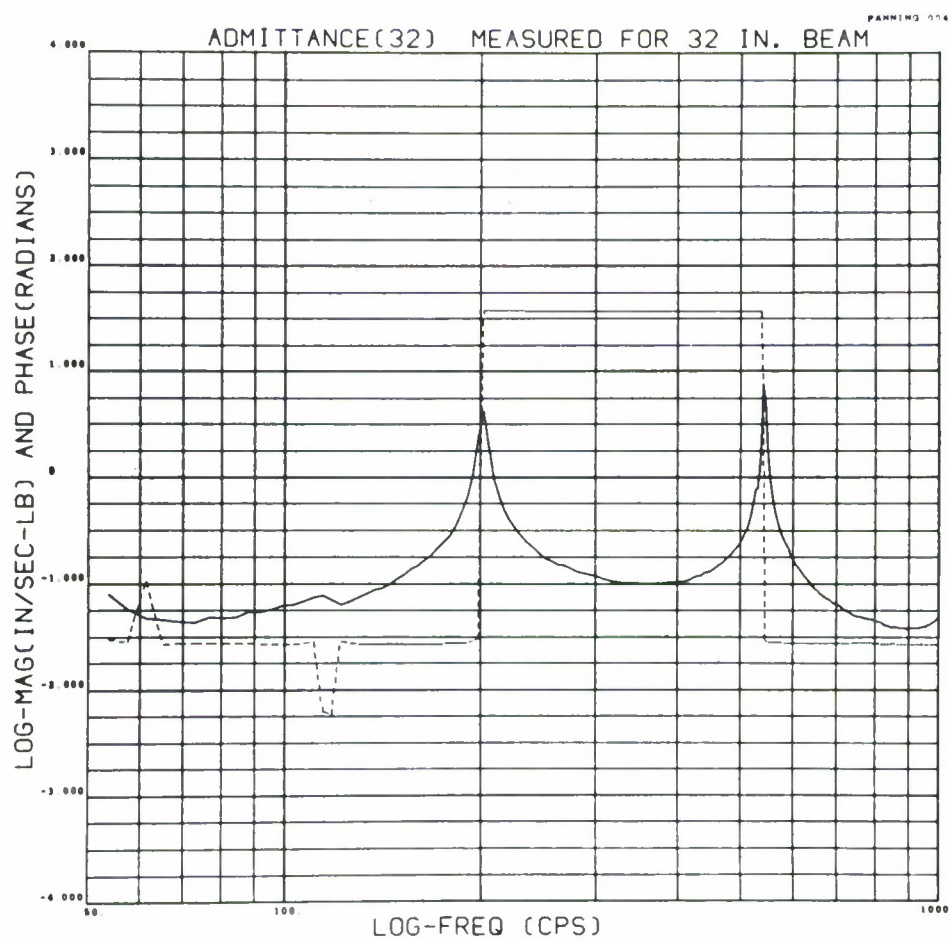
Figure 4.31

4.5 Predicting the Admittances, Resonant Frequencies and Mode Shapes for a 42-Inch Composite Beam

The admittances of a 42-inch composite beam are predicted from admittance measurements made on a 32-inch beam. The 32-inch beam has a channel cross-section and is made of steel.
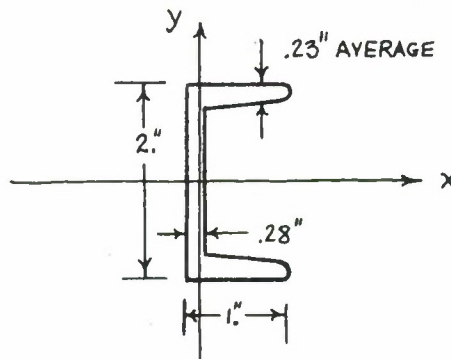
Half Scale Cross Section of the Channel

Figure 4.32

I_{yy} - Moment of Area .079 in.^{4}

A - Cross-Sectional Area .751 in.^{2}

E - Young's Modulus 29.8×10^6 psi

ρ - Density .284 $\frac{lb.}{in^3}$

L - Length 32.0 in.

Physical Properties of a 32-Inch Beam

Table 4.4

Admittance measurements are made at three locations on the 32-inch beam in the X direction as shown in Figure 4.33.

In general, nine admittances, three mutual admittances and
six cross admittances, are needed to construct a 3×3 sub-
structure admittance matrix. Because the admittance matrix
is symmetric, one needs to measure only six admittances.
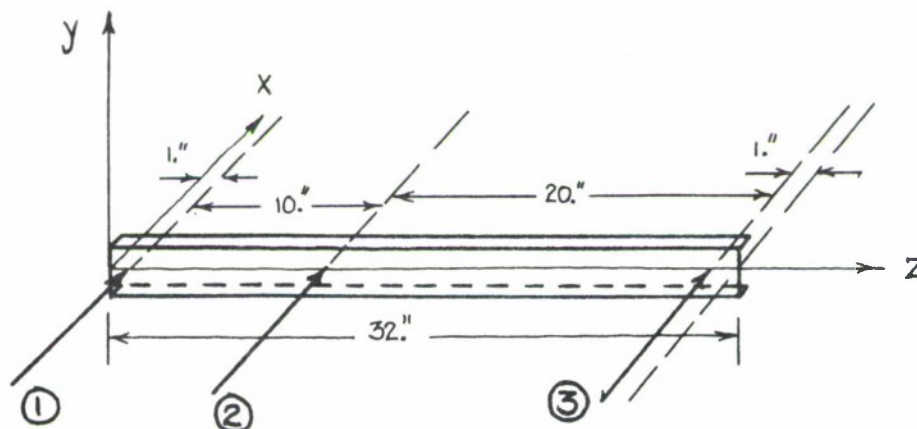
The Locations of the Measured Admittances for a 32-Inch

Beam

Figure 4.33

The mutual admittance at location 1 is the same as the
mutual admittance at location 3, so the number of admittances
that need to be measured is reduced to five. The five
measured and sampled admittances are in Figures 4.27 through
4.31. Each admittance is sampled for 249 frequencies be-
tween 50 c.p.s. and 1000 c.p.s.

Two identical substructure admittance matrices are con-
structed from the measured admittances of the 32-inch beam.
The substructure admittance matrices are inverted to sub-
structure impedance matrices which are assembled into a
composite structure impedance matrix as shown in section 2.2.

The composite structure is shown schematically in Figure 4.34.
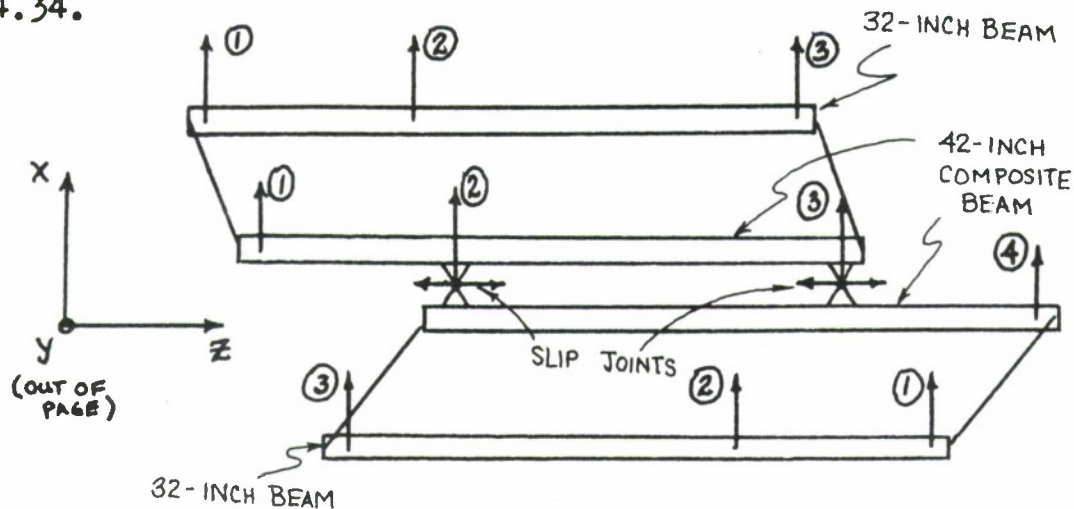
Assembling a 42-Inch Composite Beam

from Two 32-Inch Free Beams

Figure 4.34

The composite structure is made of two beams pinned together at two points. At these two points of attachment the beams have compatible X displacements; however, the beams rotate independently about the Y axis at these two points. Compatibility of displacement in the Z direction at the attachment points is not necessary, since the composite structure has no degrees of freedom in the Z direction.

The composite structure impedance matrix is inverted to a composite structure admittance matrix and the mode shapes and resonant frequencies of the composite structure are estimated as shown in section 2.4. The unique elements of the composite structure admittance matrix appear in Figures 4.35 through 4.40.

The predicted resonant frequencies and mode shapes for

Figure 4.35

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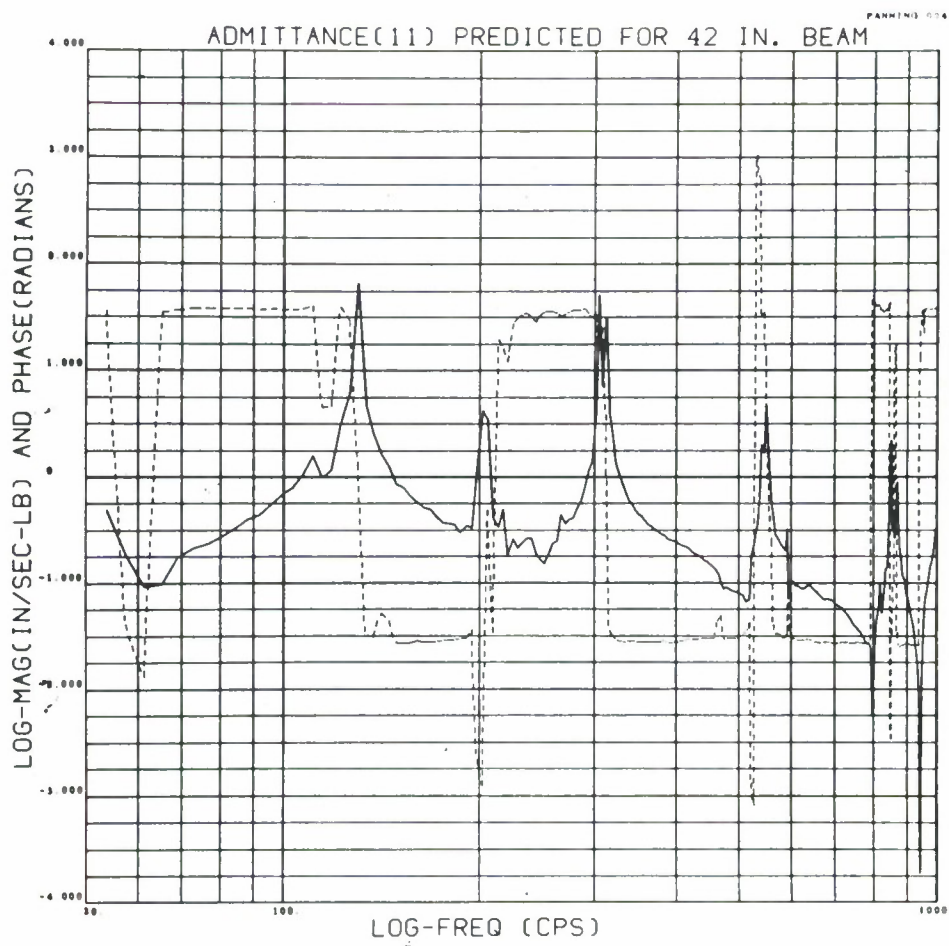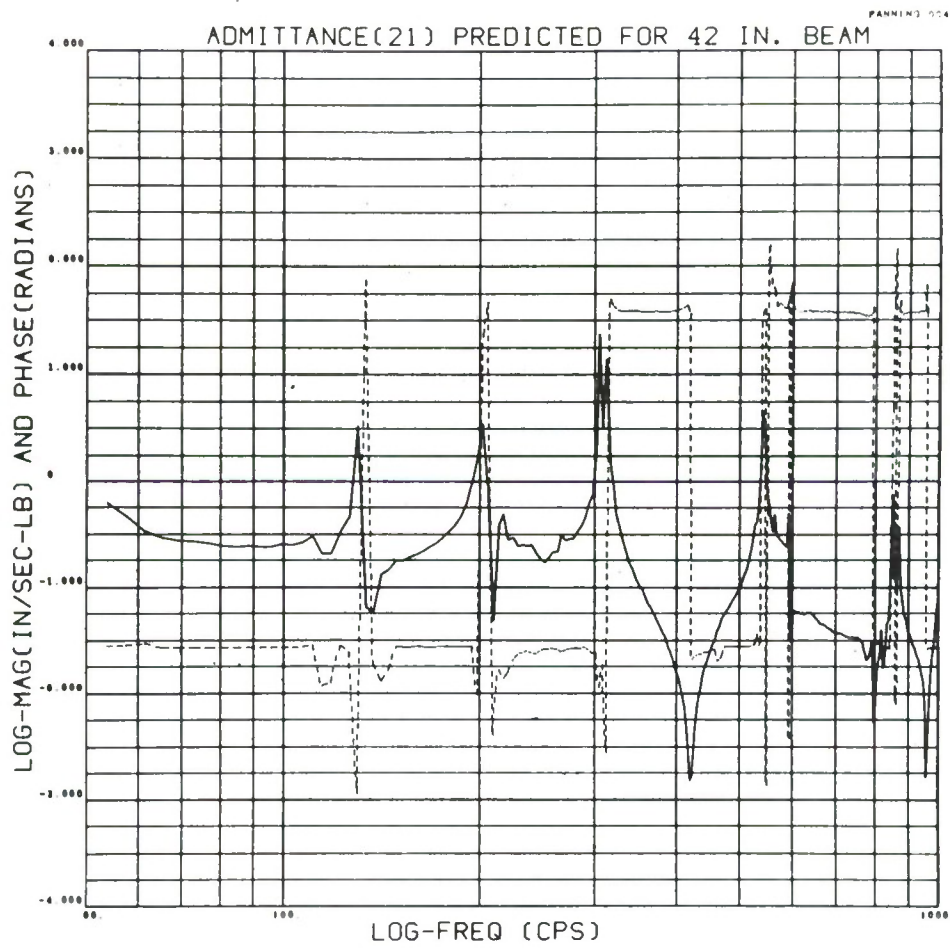
Figure 4.36

Figure 4.37

94
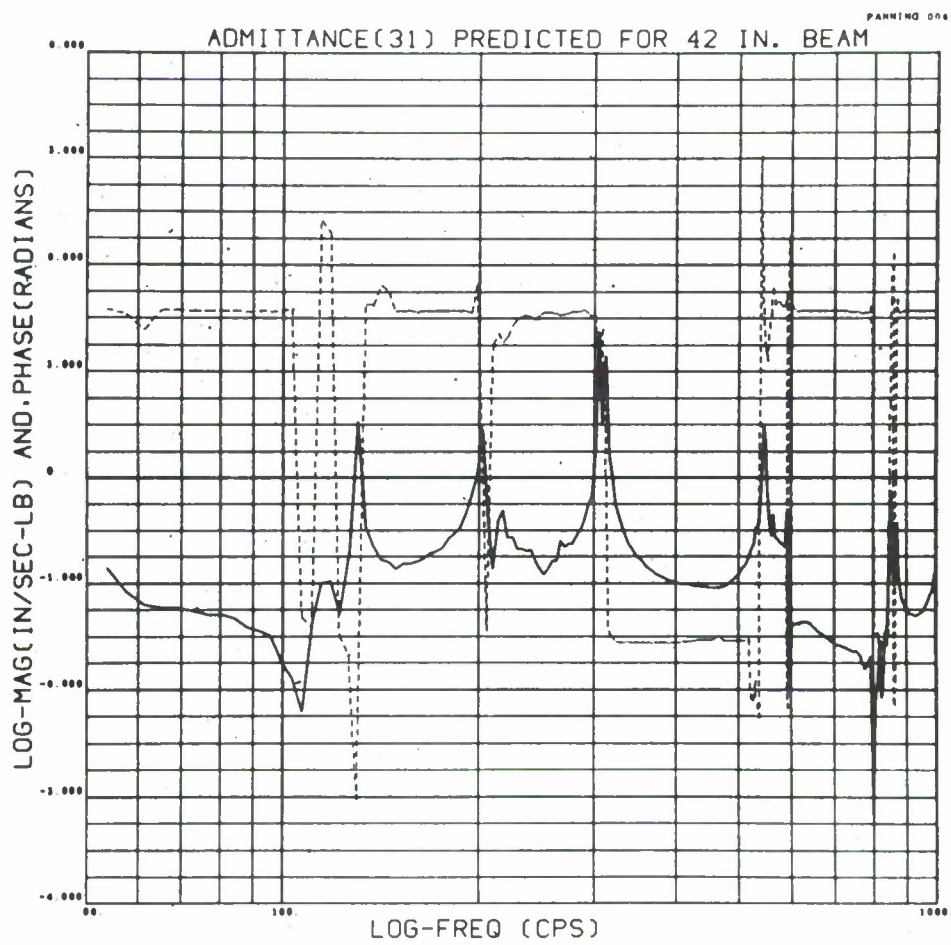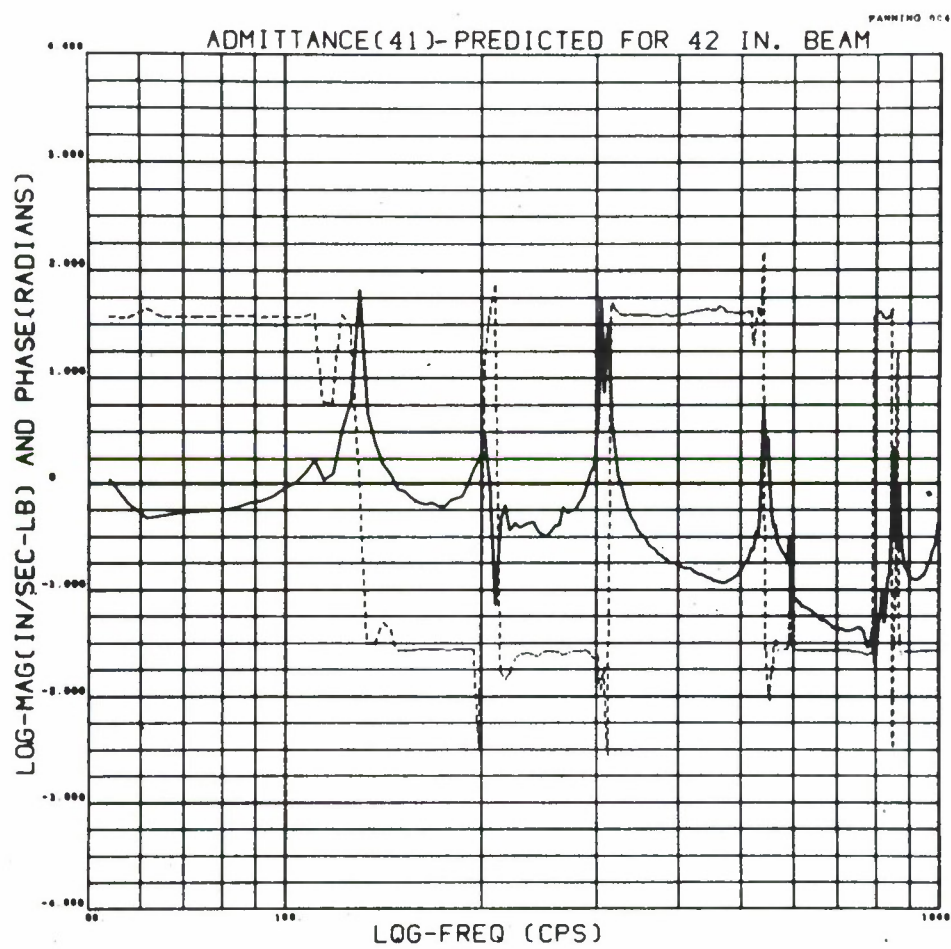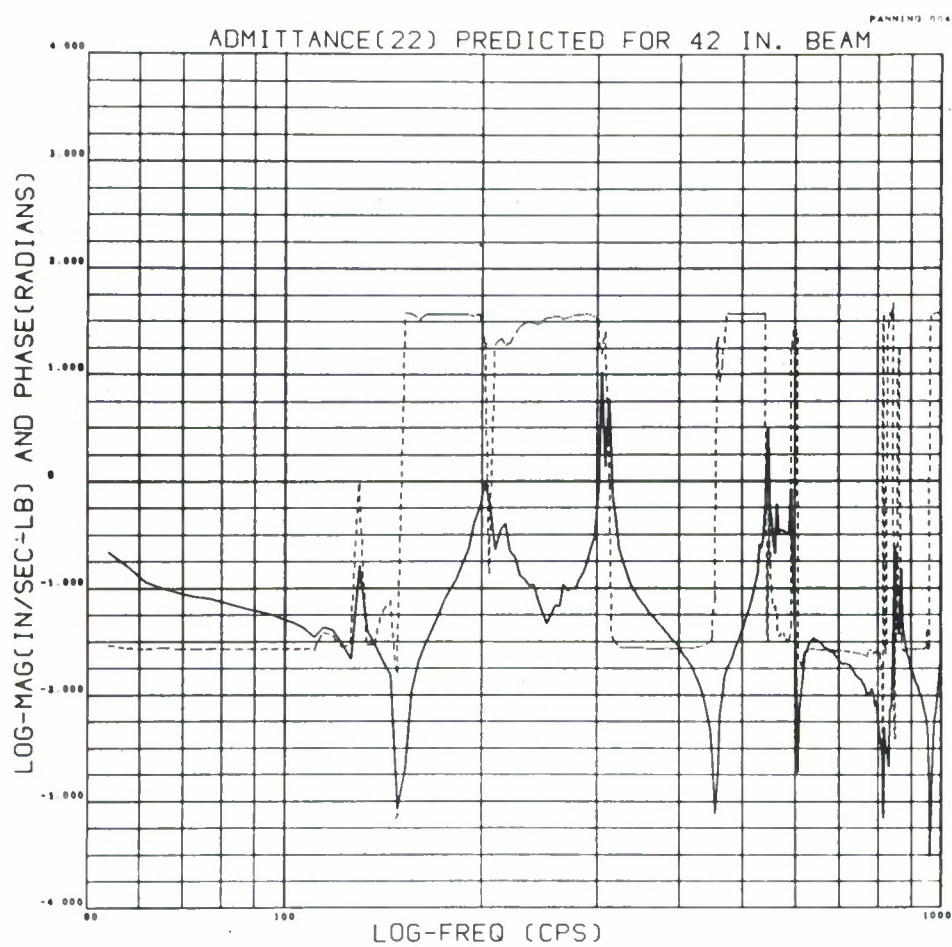
Figure 4.38

Figure 4.39
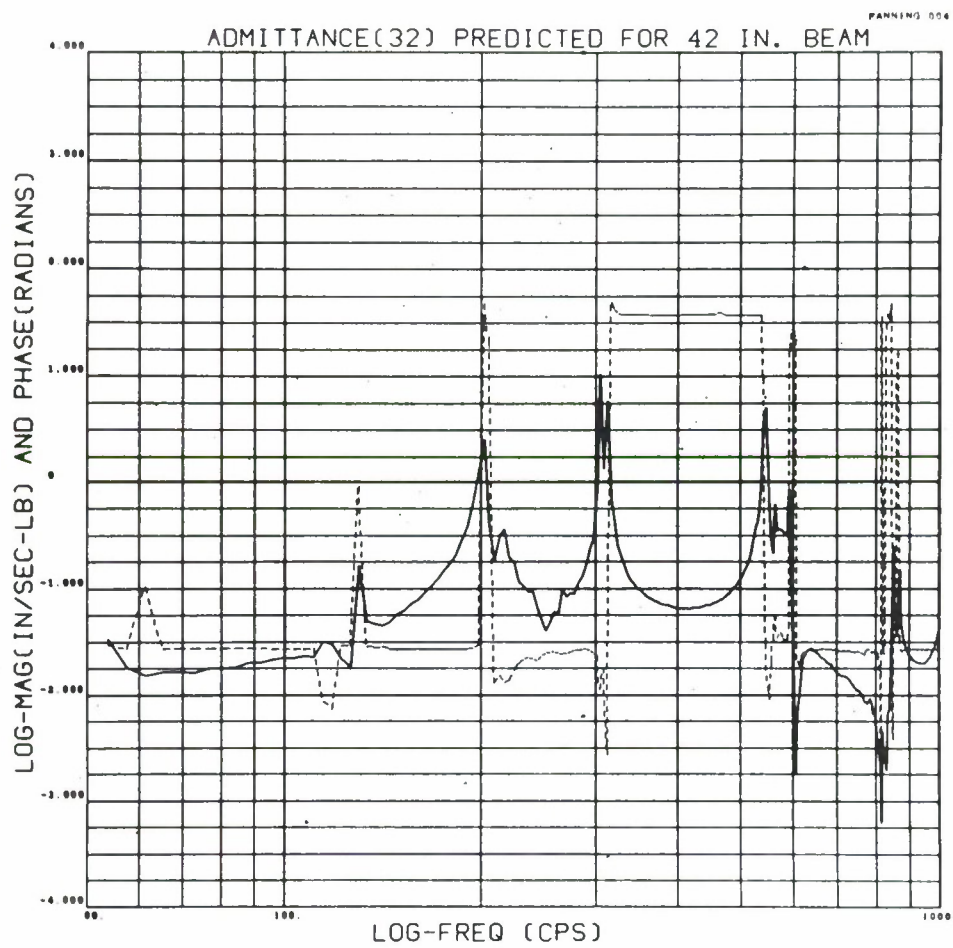
Figure 4.40

the composite structure are compared to resonant frequencies
and mode shapes calculated for the composite structure.
The calculated data is generated by a finite-element com-
puter program, STRUDL II, developed by the MIT Civil Engi-
neering Department.[10][11] A listing of the predicted and cal-
culated data appears in Table 4.5.

Two of the predicted resonant frequencies of the com-
posite structure are very close to resonant frequencies of
the substructures. As shown in section 4.3 the predicted
mode shapes for these two resonant frequencies of the com-
posite structure should not be accurate due to the distortion
of the measured admittances at the substructure resonant
frequencies. The two resonant frequencies of the composite
structure that are near resonant frequencies of the sub-
structures correspond to the second and fourth mode shapes
in Table 4.5.

Equation 4.39 shows that the effect of the suspension
system on the admittance measurements should decrease with
increasing frequency of excitation. If the suspension
system is responsible for the inaccuracy of the second and
fourth predicted mode shapes, the second mode shape should
be less accurate than the fourth mode shape. The data in
Table 4.5 shows that the second predicted mode shape at 202
c.p.s. is less accurate than the fourth predicted mode shape
at 549 c.p.s.

The calculated resonant frequencies for the composite
structure are, in general, higher than the resonant frequen-

Calculated and Predicted Mode Shapes and Resonant Frequencies for a 42-Inch Composite Beam

Table 4.5

Location	1	2	3	4	Resonant Frequency (c.p.s.)
Admittance*	11	21	31	41	
Mode Shape					
1 Predicted	1.0	-.052	-.052	1.0	130.
Calculated	1.0	-.059	-.059	1.0	147.
2 Predicted	-1.0	-.828	.669	-.715	202.**
Calculated	-1.0	.676	-.676	1.0	205.
3 Predicted	-.998	.454	-.453	1.0	305.
Calculated	-1.0	.496	-.496	1.0	415.
4 Predicted	-.893	.839	.578	-1.0	549.**
Calculated	-.986	1.0	1.0	-.986	567.
5 Predicted	.996	-.325	-.330	1.0	851.
Calculated	1.0	.178	.178	1.0	1022.
6 Predicted	-.971	.391	.406	-1.0	870.
Calculated	-1.0	.294	-.294	1.0	1037.

* The admittances are used only for the predicted data.

** These frequencies are near resonant frequencies of the 32-inch beam substructure.

cies predicted by the admittance data. The composite struc-
ture is modeled using the assumed displacement method which
should yield calculated resonant frequencies that are upper
bounds for the resonant frequencies of the real structure.

4.6 Conclusions

The resonant frequencies and mode shapes of a composite
structure can be predicted using admittance measurements
made on the constituent substructures if certain conditions
are met. First, the shaker must be able to produce enough
force to measure the admittances of the substructures over
the frequency range of interest. Second, the composite must
not have resonant frequencies that are very close together.
Third, the suspension system must not alter the admittance
measurements on the substructures. Finally, the composite
structure must not have resonant frequencies that coincide
with resonant frequencies of the constituent substructures.

Appendix A

The method of matrix inversion used in the computer program ADMIT is "triangular decomposition."[12] This method of matrix inversion entails decomposing a square symmetric matrix $[Y]$ into the matrix product of a matrix $[D]$ and its transpose.

$$[Y] = [D][D]^T \tag{A.1}$$

The matrix $[D]$ is constructed such that all elements to the right of the principal diagonal are zero. There are the same number of unique elements in $[D]$ as there are unique elements in $[Y]$. The elements of $[D]$ are solved for using equations A.2 through A.5.

$$d_{11} = Y_{11}^{1/2} \tag{A.2}$$

$$d_{n1} = Y_{n1}/d_{11} \tag{A.3}$$

$$d_{nn} = \left\{ Y_{nn} - \sum_{j=1}^{n-1} d_{nj}^2 \right\}^{1/2} \tag{A.4}$$

$$d_{mn} = \left\{ Y_{mn} - \sum_{j=1}^{n-1} d_{mj} d_{jn} \right\} / d_{nn} \quad m > n \tag{A.5}$$

The inverse of $[Y]$ is written as a product of the inverse of $[D]$ and the transpose of the inverse of $[D]$.

$$[Y]^{-1} = [D]^{T^{-1}} [D]^{-1} \tag{A.6}$$

$$[E] = [D]^{-1} \tag{A.7}$$

The elements of the matrix $[E]$ are solved for using the definition of an inverse matrix, equation A.8.

$$[D][D]^{-1} = [D][E] = [I] \tag{A.8}$$

Since all of the elements of $[D]$ to the right of its principal diagonal are zero, it can be shown that all of the elements of $[E]$ to the right of the principal diagonal are zero. Given all the elements of $[D]$, one can solve for the elements in $[E]$ using equations A.9 and A.10.

$$e_{nn} = d_{nn}^{-1} \tag{A.9}$$

$$e_{mn} = \left\{ -\sum_{j=n}^{m-1} d_{mj} e_{jn} \right\} / d_{nn} \qquad m > n \tag{A.10}$$

Appendix B

Computer Program Listings

```
CP67USERIC  FANNING    05/04/71    15C5B046    FANNING    ************************    AOM00010
                                                                                      AOMODD20
                                                                                      ADMDD030
C          THE CCMPUTER PROGRAM  ADMIT   PREOICTS THE ADMITTANCE MATRIX              AOM00040
C     FOR A COMPOSITE STRUCTURE FROM THE ADMITTANCE MATRICES CF THE                  ADM00050
C     CONSTITUENT SUBSTRUCTURES.  THE SUBSTRLCTURE ACMITTANCE MATRICES               AOM00060
C     ARE READ INTO THE MATRIX  SUBAD  AND ARE INVERTEO TC SLBSTRUCTURE              AOM00070
C     IMPEDANCES. THE SUBSTRUCTURE IMPEDANCE MATRICES ARE ASSEMBLED                  AOM000B0
C     INTO A COMPOSITE STRUCTURE IMPEOANCE MATRIX AS SHOWN IN SECTION                AOM009D
C     2.2.  THE COMPOSITE STRUCTURE IMPEDANCE MATRIX IS INVERTED TO                  AOMDD100
C     THE COMPOSITE STRUCTURE ADMITTANCE MATRIX, TOTAO . THE PREOICTEOAOMOO110
C     ADMITTANCE MATRICES FOR DIFFERENT FREQUENCIES CF EXCITATION ARE                AOMO12D
C     STORED IN THE MATRIX  OUTPUT .                                                 AOMO013D
C          ALL ACMITTANCE AND IMPEOANCE MATRICES ARE STORES IN ONE                   AOMD0140
C     DIMENSIONAL ARRAYS.  THE ELEMENTS CF IMPEDANCE AND ACMITTANCE                  AOMO0150
C     MATRICES THAT ARE TO THE RIGHT OF THE PRINCIPAL DIAGCNAL ARE NCT               AOMO0160
C     USED BECAUSE THE MATRICES ARE SYMMETRIC.  THE INTEGER FUNCTION                 AOMO0170
C     IOENT  IS USEC TO ADDRESS THE ELEMENTS OF THE SQUARE IMPEDANCE                 AOMO0180
C     AND ADMITTANCE MATRICES IN ONE DIMENSICNAL ARRAYS.                            AOMOD190
C                                                                                    AOMOD20D
C                                                                                    AOM00210
C                                                                                    AOM00220
C     SM  IS THE NUMBER CF SUBSTRUCTURE ADMITTANCE MATRICES USEC TO                  AOM00230
C     CALCULATE THE COMPOSITE STRUCTURE ADMITTANCE MATRIX.                           AOM00240
C                                                                                    AOM00250
C     OSM  IS THE DIMENSION OF THE LARGEST SUBSTRUCTURE ADMITTANCE                    AOM00260
C     MATRIX                                                                         AOM00270
C                                                                                    AOM00280
C     OTM  IS THE DIMENSION OF THE  COMPOSITE STRUCTURE ADMITTANCE                   AOM00290
C     MATRIX                                                                         AOM00300
C                                                                                    AOM00310
C     OP  IS THE NUMBER DF FREQUENCIES FOR WHICH SUBSTRUCTURE ACMIT-                 ADM00320
C     TANCE MATRICES HAVE BEEN CONSTRUCTEO.                                          AOM00330
C                                                                                    AOM00340
C     FBL  AND  FBU  ARE THE LOWER AND UPPER BCUNDS FOR THE FREQUENCY                AOM00350
C     SWEEP THAT GENFRATEO THE ADMITTANCE DATA.                                      AOMOD360
C                                                                                    AOMOD370
C     IDIS  IS THE MATRIX THAT CONTAINS THE NODE NUMBERS FOR THE                     AOM00380
C     COMPOSITE STRUCTURE FOR EACH CF THE NODES ON THE CONSTITUENT                   AOM00390
C     SUBSTRUCTURES.                                                                 AOM0D400
C                                                                                    AOM00410
      DIMENSION SUBAD(6),TOTAO(IO),IDIS(I3,3),OATA(249,2,6,,2)                       AOM00420
      DIMENSIDN OUTPUT(2,10,249)                                                     AOM00430
      COMPLEX SUBAD,TOTAO,TOTA,CTOTAD,STORE,CTORE                                    AOM00440
   IO FORMAT(E2D.1C)                                                                 AOM00450
   2D FORMAT(4E19.1D)                                                                AOM00460
 1DD1 FORMAT(F1D.D)                                                                  AOM00470
 1DD5 FORMAT(F8.2)                                                                   AOM00480
 1DD2 FORMAT(2IE20.1n,5X))                                                           AOM00490
 1000 FORMAT(9I1)                                                                    AOM00500
C                                                                                    AOM00510
```

```
C                                                                       ADM00520
C                                                                       ADM00530
      THE INITIALIZATION DATA IS READ HERE.                             ADM00540
C                                                                       ADM00550
C                                                                       ADM00560
                                                                        ADM00570
      READ(5,100I) SM,DSM,DTM,DP,F8L,F8U                                ADM00580
      NSM=SM                                                            ADM00590
      IDSN=DSM                                                          ADM00600
      IDTN=DTM                                                          ADM00610
      IDP=DP                                                            ADM00620
      IDSM=(IDSN*(IDSN+1))/2                                            ADM00630
      IDTM=(IDTN*(IDTN+1))/2                                            ADM00640
      FOI=(FHU-F8L)/DP                                                  ADM00650
C                                                                       ADM00660
C                                                                       ADM00670
C                                                                       ADM00680
      THE SUBSTRUCTURE ADMITTANCE MATRICES FCR ALL FREQUENCIES CF       ADM00690
      EXCITATION ARE READ INTO THE MATRIX DATA HERE.                    ADM00700
C                                                                       ADM00710
C                                                                       ADM00720
C                                                                       ADM00730
      READ(1,20)(((((DATA(M7,M6,M8,M9),M7=1,ICP),M6=1,2),M8=1,IDSM),M9=1,ADM00740
     INSM)                                                              ADM00750
      READ(5,1000)(((IDIS(N,M),N=1,IDSN),M=1,NSM)                       ADM00760
      DO 300 M7=1,IDP                                                   ADM00770
      DO 303 M11=1,IDTM                                                 ADM00780
      TOTAD(M11)=(0.,0.)                                                ADM00790
  303 CONTINUE                                                          ADM00800
C                                                                       ADM00810
C                                                                       ADM00820
C                                                                       ADM00830
      THE PHASE AND MAGNITUDE DATA FOR THE ADMITTANCE IS USED TC        ADM00840
      CALCULATE CORRESPONDING COMPLEX NUMBERS.                          ADM00850
C                                                                       ADM00860
C                                                                       ADM00870
C                                                                       ADM00880
      DO 214 M9=1,NSM                                                   ADM00890
      DO 301 M8=1,IDSM                                                  ADM00900
      RMAG=DATA(M7,1,M8,M9)                                             ADM00910
      PHASE=DATA(M7,2,M8,M9)                                            ADM00920
      PH1=ABS(PHASE-1.5707)                                             ADM00930
      PH2=ABS(PHASE+1.5707)                                             ADM00940
      IF(PH1.LT..5) PHASE=1.5707+.1*PH1                                 ADM00950
      IF(PH2.LT..5) PHASE=-1.5707+.1*PH2                                ADM00960
      C1=COS(PHASE)                                                     ADM00970
      S1=SIN(PHASE)                                                     ADM00980
      SUBAO(M8)=(1.,0.)*C1+(0.,1.)*S1                                   ADM00990
      SUBAO(M8)=(10.**RMAG)*SUBAO(M8)                                   ADM01000
  301 CONTINUE                                                          ADM01010
C                                                                       ADM01020
C                                                                       ADM01030
C                                                                       ADM01040
      THE SUBSTRUCTURE ADMITTANCE MATRICES ARE INVERTED TC SUB-         ADM01050
```

```
C
C      STRUCTURE IMPEDANCE MATRICES.                                          ADM01060
C                                                                            ADM01070
                                                                             ADM01080
       DO 205 N=1,IDSN                                                       ADM01090
       N1=N-1                                                                ADM01100
       IF(N1.EQ.0) GO TO 204                                                 ADM01110
       DO 202 I=1,N1                                                         ADM01120
       I1=I-1                                                                ADM01130
       STORE=(0.,0.)                                                         ADM01140
       IF(I.EQ.1) GO TO 201                                                  ADM01150
       DO 200 M=1,I1                                                         ADM01160
       STORE=STORE+SUPAC(IDENT(N,M))*SUBAD(IDENT(I,M))                       ADM01170
  200  CONTINUE                                                              ADM01180
  201  SUBAD(IDENT(N,I))=(SUBAD(IDENT(N,I))-STORE)/SUBAD(IDENT(I,I))         ADM01190
  202  CONTINUE                                                              ADM01200
       STORE=(0.,0.)                                                         ADM01210
       DO 203 M=1,N1                                                         ADM01220
       STORE=STORE+SUPAD(IDENT(N,M))**2                                      ADM01230
  203  CONTINUE                                                              ADM01240
  204  IF(N1.EQ.0)STORE=(0.,0.)                                              ADM01250
       SUBAD(IDENT(N,N))=CSQRT(SUBAD(IDENT(N,N))-STORE)                      ADM01260
  205  CONTINUE                                                              ADM01270
       DO 207 N=1,IDSN                                                       ADM01280
       N1=N-1                                                                ADM01290
       IF(N1.EQ.0) GO TO 210                                                 ADM01300
       DO 208 I=1,N1                                                         ADM01310
       I1=I-1                                                                ADM01320
       STORE=(0.,0.)                                                         ADM01330
       DO 209 M=I,N1                                                         ADM01340
       STORE=STORE-SUPAC(IDENT(I,M))*SUBAC(IDENT(N,M))                       ADM01350
  209  CONTINUE                                                              ADM01360
       SUBAD(IDENT(N,I))=STORE/SUBAD(IDENT(N,N))                             ADM01370
  208  CONTINUE                                                              ADM01380
  210  SUBAD(IDENT(N,N))=(1.,0.)/SUBAD(IDENT(N,N))                           ADM01390
  207  CONTINUE                                                              ADM01400
       DO 211 N=1,IDSN                                                       ADM01410
       N1=N-1                                                                ADM01420
       DO 212 I=1,N                                                          ADM01430
       STORE=(0.,0.)                                                         ADM01440
       DO 213 M=N,IDSN                                                       ADM01450
       STORE=STORE+SUPAD(IDENT(M,N))*SUBAD(IDENT(M,I))                       ADM01460
  213  CONTINUE                                                              ADM01470
       SUBAD(IDENT(N,I))=STORE                                               ADM01480
  212  CONTINUE                                                              ADM01490
  211  CONTINUE                                                              ADM01500
C                                                                            ADM01510
C                                                                            ADM01520
C                                                                            ADM01530
C      THE SUBSTRUCTURE IMPEDANCES ARE ASSEMBLED INTO A COMPOSITE            ADM01540
C      STRUCTURE IMPEDANCE MATRIX.                                           ADM01550
C                                                                            ADM01560
C                                                                            ADM01570
```

```
C
C
          DO 216 I1=1,(D*N                                    ADMD1580
          DO 217 J1=1,I1                                      ADMD1590
          N2=IDIS(I1,M9)                                      ADMD1600
          M2=IDIS(J1,M9)                                      ADMD1610
          TOTAD(IDENT(N2,M2))=SUBAD(IDENT(I1,J1))+TCTAD(IDENT(N2,M2))   ADMD1620
  217     CONTINUE                                            ADMD1630
  216     CONTINUE                                            ADMD1640
  214     CONTINUE                                            ADMD1650
C                                                             ADMD1660
C                                                             ADMD1670
C                                                             ADMD1680
C             THE COMPOSITE STRUCTURE IMPEDANCE MATRIX IS INVERTED TO A   ADMD1690
C         COMPOSITE STRUCTURE ADMITTANCE MATRIX.              ADMD1700
C                                                             ADMD1710
C                                                             ADMD1720
C                                                             ADMD1730
C                                                             ADMD1740
          DO 1D5 N=1,IDTN                                     ADMD1750
          N1=N-1                                              ADMD1760
          IF(N1.EQ.D) GO TO 1D4                               ADMD1770
          DO 1D2 I=1,N1                                       ADMD1780
          I1=I-1                                              ADMD1790
          STORE=(0.,0.)                                       ADMD1800
          IF(I.EC.1) GC TO 1D1                                ADMD1810
          DO 1DD M=1,I1                                       ADMD1820
          STCRE=STCRE+TOTAC(IDENT(N,M))*TDTAC(IDENT(I,M))     ADMD1830
  100     CONTINUE                                            ADMD1840
  101     TOTAD(IDENT(N,I))=(TOTAD(IDENT(N,1))-STCRE)/TOTAD(IDENT(I,1))  ADMD1850
  1D2     CONT(NUE                                            ADMD1860
          STORE=(0.,0.)                                       ADMD1870
          DO 1D3 M=1,N1                                       ADMD1880
          STORE=STORE+TOTAD(ICENT(N,M))**2                    ADMD1890
  1D3     CONTINUE                                            ADMD19DD
  1D4     IF(N1.EQ.D)STORE=(0.,D.)                            ADMD1910
          TOTAC(IDENT(N,N))=CSCRT(TOTAD(IDENT(N,N))-STORE)    ADMD1920
  1D5     CONTINUE                                            ADMD1930
          DO 107 N=1,IDTN                                     ADMD1940
          N1=N-1                                              ADMD1950
          IF(N1.EQ.D) GO TO 11D                               ADMD1960
          DO 108 I=1,N1                                       ADMD197D
          I1=I-1                                              ADMD1980
          STORE=(0.,D.)                                       ADMD1990
          DO 1D9 M=1,N1                                       ADMD2000
          STCRE=STORE-TOTAC(IDENT(I,M))*TOTAD(IDENT(N,M))     ADMD2D1D
  109     CONT(NUE                                            ADMD2D2D
          TDTAD(IDENT(N,I))=STORE/TOTAD(IDENT(N,N))           ADMD2030
  1D8     CONTINUE                                            ADMD2040
  110     TOTAD(IDENT(N,N))=(1.,D.)/TOTAD(IDENT(N,N))         ADMD2050
  107     CONTINUE                                            ADMD2060
          DO 111 N=1,IDTN                                     ADMD2070
          N1=N-1                                              ADMD2080
          DO 112 I=1,N                                        ADMD2090
          STORE=(0.,0.)                                       ADMD2100
                                                              ADMD2110
```

```
      DO 113 M=N,ICTN                                                   AOM02120
      STORE=STORE+TOTAC((DENT(M,N))*TDTAD(IOENT(M,I))                   AOM02130
  113 CONTINUE                                                          AOM02140
      TOTAO(IDENT(N,I))=STORE                                           AOM02150
  112 CONTINUE                                                          AOM02160
  111 CONTINUE                                                          AOM02170
      FQ=M7*FQ(I+FBL                                                    AOM02180
C                                                                       AOM02190
C                                                                       AOM02200
C     THE COMPLEX NUMBERS CORRESPONDING TO THE PREDICTED ADM(T-         AOM02210
C     TANCES OF THE COMPCSITE ARE USED TO CALCULATE THE PHASE AND       AOM02220
C     MAGNITUDE OF THE PREDICTED ADMITTANCES                            AOM02230
C                                                                       AOM02240
C                                                                       AOM02250
C                                                                       AOM02260
C                                                                       AOM02270
      DO 302 M=1,ICTM                                                   AOM02280
      TOTA=TOTAD(M)                                                     AOM02290
      CTOTAO=CCNJG(TOTA)                                                AOM02300
      RMAG=CTOTAD*TOTAC(M)                                              AOM02310
      TOP=REAL(TOTA)                                                    AOM02320
      BUTTCM=AIMAG(TOTA)                                                AOM02330
      PHASE=ATAN2(BOTTOM,TOP)                                           AOM02340
      RMAG=ALOG10(ABS(RMAG))/2.                                         ADM02350
      OUTPUT(1,M,M7)=RMAG                                               AOM02360
      OUTPUT(2,M,M7)=PHASE                                              AOM02370
  302 CONTINUE                                                          AOM02380
  300 CONTINUE                                                          AOM02390
C                                                                       AOM02400
C                                                                       AOM02410
C                                                                       AOM02420
C     THE PREDICTED ADMITTANCES ARE STORED IN THE ARRAY  OUTPUT .       AOM02430
C                                                                       AOM02440
C                                                                       AOM02450
C                                                                       AOM02460
      WRITE(12,201(((OUTPUT(I10,I11,I12),I12=1,IOP),(IO=1,2),I11=1,ICTM)AOM02470
      END                                                               AOM02480
C                                                                       AOM02490
C                                                                       AOM02500
C                                                                       AOM02510
      FUNCTION ICENT(M,N)                                               AOM02520
      IF(M.GE.N) ICENT=(M*(M-1))/2+N                                    AOM02530
      IF(M.LT.N) ICENT=(N*(N-1))/2+M                                    ADM02540
      RETURN                                                            AOM02550
      END                                                               AOM02560
```

```
CP67USERIO  FANNING      05/04/71    15C58049    FANNING     ********************
C                                                                          CON00010
C                                                                          CON00020
C       THE CCMPUTER PROGRAM  CONVERT   INCCRPORATES THE SUBROUTINE        CON00030
C       RADATA   TC CONVERT THE FORMAT OF THE CATA FROM THE  RACIATICN     CON00040
C       A/D CONVERTER TO A FORMAT CCMPATIBLE WITH THE I8M 360 CCMPLTER.    CON00050
C       THE PRCGRAM CONVERTS THE FORMAT CF THE CATA CNE FILE AT A TIME.    CON00060
C                                                                          CON00070
C                                                                          CON00080
C                                                                          CON00090
C                                                                          CON00100
        CALL TAPSET(1,11,4C96,3,1,4096)                                    CON00110
        DIMENSIUN STORF(2048)                                             CON00120
      3 FORMAT(' PARITY ERRCR,M=',I4)                                      CON00130
      8 FORMAT(E20.IO)                                                     CON00140
        DO 9 N=1,2D48                                                      CON00150
        STORE(N)=0.                                                       CON00160
      9 CCNTINUE                                                           CON00170
        NFILE=1                                                            CON00180
        NREC=249                                                           CON00190
        DO 7 N=1,NFILE                                                     CON00200
        DO I M=1,NREC                                                      CON00210
        CALL RADATA(1,STORE,NC)                                           CON00220
        GO TO (4,2,7),NC                                                   CON00230
      2 WRITE(2,3) M                                                       CON00240
      4 STOR=0.                                                            CON00250
        DO 5 I=1848,2048                                                   CON00260
        STOR=STOR+STORF(I)                                                 CON00270
      5 CCNTINUE                                                           CON00280
        STOR=STOR/200.                                                     CON00290
        WRITE(12,8) STOR                                                   CON00300
      1 CONTINUE                                                           CON00310
      7 CONTINUE                                                           CON00320
        END                                                               CON00330
C       FORNI JANUARY 1970   SUBROUTINE RADATA(NSKP,ARRAY,NCODE)          CON00340
C       READ, CONVERT DATA ON TAPE GENERATED BY A TC C CCNVERTERCON00350
        SUBROUTINE RADATA(NSKP,ARRAY,NCODE)                               CON00360
C       READ ONE RECORD CF 4096 CHARACTERS AND CONVERT DATA  OR          CON00370
C       SKIP TO THE END OF FILE MARK         NC DATA                      CON00380
C                                                                          CON00390
C       CONVERSICN OF DATA IS  TWO SIX BIT BINARY INTEGERS CCMBINED        CON00400
C       INTO ONE BINARY INTEGER. THE FIRST SIX BIT INTEGER READ IS         CON00410
C       THE HIGH ORDER BITS. THE INTEGER IS THEN CONVERTED TO A            CON00420
C       FLOATING POINT NUMBER.                                            CON00430
C                                                                          CON00440
C       NSKP= 1   SAME FILE   READ AND CONVERT DATA                       CON00450
C       NSKP= 2   NEXT FILE   SKIP TO NEXT FILE. DO NOT CCNVERT DATA       CON00460
C       ARRAY=    DATA ARRAY                                              CON00470
C                 ASSUMES  REAL*4 ARRAY(2048)                             CON00480
C       NCODE= I   DATA CK  CCNVERT                                       CON00490
C       NCODE= 2   PARITY ERROR DETECTED  HOWEVER   CONVERT DATA          CON00500
```

```
C           NCODE= 3   END OF FILE DETECTED                                      CON00510
C                                                                                CON00520
C           FILE AND RECORD COUNTERS                                             CON00530
      INTEGER*4 IFILE/0/, IREC/0/                                                CON00540
C           TAPE ASSIGNMENT                                                      CON00550
      INTEGER*4 IUNIT/11/                                                        CON00560
C           DATA ARRAY                                                           CON00570
      DIMENSION ARRAY(2048)                                                      CON00580
      INTEGER*2 IRRAY(2048)                                                      CON00590
      GO TO (100,200), NSKP                                                      CON00600
C           READ AND CONVERT DATA                                                CON00610
  100 READ (IUNIT,101,ERR=300,END=390)  IRRAY                                    CON00620
  101 FORMAT(21(10C(A2)))                                                        CON00630
C 100 CALL RORV(IUNIT,IRRAY,&390,&300)                                           CON00640
      CALL PDUMP(IRRAY(1),IRRAY(4),0)                                            CON00650
      NCODE= 1                                                                   CON00660
      IREC= IREC + 1                                                             CON00670
C           DATA CONVERSION                                                      CON00680
  105 DO 110 I=1,2048                                                            CON00690
      I12= IRRAY(I)                                                              CON00700
C           LEFT HALF OF INTEGER                                                 CON00710
      I1L= I12 / 256 * 256                                                       CON00720
C           RIGHT HALF OF INTEGER                                                CON00730
      I1R= I12 - I1L                                                             CON00740
C           COMPLETE INTEGER.  NEGATIVE IS ONES COMPLIMENT                       CON00750
      INT= (I1R + I1L / 4) *  16 * 2**16   16 / 2**16                            CON00760
      INT= (I1R + I1L / 4) *  104857 /  1C48576                                  CON00770
C           CHANGE ONES COMPLIMENT TO TWOS COMPLIMENT                            CON00780
      IF (INT.LT.0)  INT= INT + 1                                               CON00790
C           REPLACE INPUT VALUE BY CALCULATED INTEGER                            CON00800
  110 ARRAY(I)= INT                                                              CON00810
      GO TO 500                                                                  CON00820
C           SKIPPING TO END OF FILE                                             CON00830
  200 READ (IUNIT,210,ERR=200,END=400)  NSKIP                                    CON00840
C 200 READ (IUNIT,210,ERR=200,END=395)  NSKIP                                    CON00850
  210 FORMAT(1A1)                                                                CON00860
      IREC= IREC + 1                                                             CON00870
      GO TO 200                                                                  CON00880
C           PARITY ERROR DETECTED ON TAPE                                        CON00890
  300 NCODE= 2                                                                   CON00900
      IREC= IREC + 1                                                             CON00910
      WRITE (6,310) IREC                                                         CON00920
  310 FORMAT(' PARITY ERROR DETECTED.   SKIPPED RECORD',I5)                      CON00930
C 310 FORMAT(' PARITY ERROR DETECTED.   RECORD',I5)                             CON00940
      READ (IUNIT,210) NSKIP                                                     CON00950
      GO TO 100                                                                  CON00960
      GO TO 105                                                                  CON00970
C           END OF FILE DETECTED                                                 CON00980
  390 WRITE (6,391)                                                              CON00990
  391 FORMAT('  READ AN END OF FILE')                                            CON01000
C 391 FORMAT(' RORV READ AN END OF FILE')                                        CON01010
      GO TO 400                                                                  CON01020
C                                                                                CON01030
C 395 READ (IUNIT,210) NSKIP                                                     CON01040
```

```
C     BACKSPACE IUNIT                                              CON01050
400   NCCCE = 3                                                    CON01060
      IFILE= IFILE + 1                                             CON01070
      WRITE (6,410) IREC,IFILE                                     CON01080
410   FORMAT(' END OF FILE DETECTED AFTER RECORD',I5,',    FCR FILE',I3) CON01090
      IREC= C                                                      CON01100
500   RETURN                                                       CON01110
      ENO                                                          CON01120
```

```
CP67USERIC  FANNING      05/04/71    15C58051     FANNING     ********************  SCA00010
                                                                                    SCA00020
C                                                                                   SCA00030
C         THE PROGRAM SCALE  IS INTENDEO TC BE USEO ON A TIME                       SCA00040
C     SHARING SYSTEM TC SCALE THE CONVERTEO DIGITAL DATA FROM                       SCA00050
C     THE RADIATICN A/D CONVERTER. THE METHCO CF SCALING IS                         SCA00060
C     EXPLAINED IN CHAPTER 3 OF THIS PAPER.                                          SCA00070
C                                                                                   SCA00080
C         THE PROCECURE FOR USING THE PROGRAM IS AS FOLLCWS.                         SCA00090
C                                                                                   SCA00100
C     1.  FROM THE CONSOLE PUT IN THE VALUES OF THE LOWER ANC UPPER                  SCA00110
C     BOUNDS OF THE FREQUENCY SWEEP AS WELL AS THE THE NUMBER CF DATA                SCA00120
C     POINTS FOR WHICH THE DATA IS SAMPLEO.                                          SCA00130
C                                                                                   SCA00140
C     2.  ENTER THE SUBSCRIPTS OF TWO FREQUENCIES FOR WHICH THE DATA                 SCA00150
C     HAS BEEN SAMPLEO.                                                              SCA00160
C                                                                                   SCA00170
C     3.  THE FRECUENCIES THAT HAVE BEEN CHCSEN WILL BE TYPEO AT THE                 SCA00180
C     CONSCLE.                                                                       SCA00190
C                                                                                   SCA00200
C     4.  IF YOU CO NOT WANT TO USE DATA CORRESPCNOING TC THESE TWO                  SCA00210
C     FREQUENCIES TO SCALE ALL THE DATA TYPE IN A  0 , ANO REPEAT                    SCA00220
C     STEPS 2 ANO 3.                                                                 SCA00230
C                                                                                   SCA00240
C     4.  IF THE FRFQUENCIES ARE ACCEPTABLE THEN TYPE A  1  FCR A                    SCA00250
C     MAGNITUOE SCALING OR A  2  FOR A PHASE SCALING.                                SCA00260
C                                                                                   SCA00270
C     5.  TYPE IN THE VALUES OF THE MAGNITUCE OR PHASE FRCM THE                      SCA00280
C     ANALOG PLOT FOR THE TWO FREQUENCIES ANC THE ENTIRE SEF OF DATA                 SCA00290
C     IS SCALEC AS SHCWN IN CHAPTER 3.                                               SCA00300
C                                                                                   SCA00310
C                                                                                   SCA00320
C                                                                                   SCA00330
C     FBL  ANO  FBU ARE THE LOWER ANO UPPER BCUNOS OF THE FREQUENCY                  SCA00340
C     SWEEP THAT GENFRATED THE AOMITTANCE OATA.                                      SCA00350
C                                                                                   SCA00360
C     OP  IS THE NUMBER OF DATA POINTS IN THE ENTIRE OATA SET.                       SCA00370
C                                                                                   SCA00380
C     SP1 ANO  SP2  ARE THE SUBSCRIPTS OF THE FREQUENCIES FCR WHICH                  SCA00390
C     DIGITAL DATA IS COMPAREO TO PLOT OATA.                                         SCA00400
C                                                                                   SCA00410
C     RMAG1  AND  RMAG2  ARE THE VALUES OF PLOT OATA REAO IN FRCM THE                SCA00420
C     CONSCLE.                                                                       SCA00430
C                                                                                   SCA00440
C                                                                                   SCA00450
C                                                                                   SCA00460
C                                                                                   SCA00470
      OIMENSION OATA(249)                                                           SCA00480
      REAO(5,100)FBL,FBU,CP                                                         SCA00490
  100 FORMAT(FIO.O)                                                                 SCA00500
      IOP=OP
```

```
      READ(11,106)(DATA(M),M=1,IDP)           SCA00510
102   FORMAT(E20.10)                          SCA00520
1     READ(5,100)SP1,SP2                      SCA00530
      ISP1=SP1                                SCA00540
      ISP2=SP2                                SCA00550
      FQI=(F8U-FBL)/CP                        SCA00560
      FQI=FQI*SP1+F8L                         SCA00570
      FO2=FQI*SP2+F8L                         SCA00580
      WRITE(6,103) FQI,FQ2                    SCA00590
103   FORMAT(2(F8.2,5X))                      SCA00600
104   READ(5,104) ICHECK                      SCA00610
      FORMAT(I1)                              SCA00620
      IF(ICHECK.EQ.0) GO TO 1                 SCA00630
      IF(ICHECK.EQ.2) GO TO 4                 SCA00640
      READ(5,105)RMAG1,RMAG2                  SCA00650
105   FORMAT(2F8.4)                           SCA00660
      RMAG1=ALOG10(RMAG1)                     SCA00670
      RMAG2=ALOG10(RMAG2)                     SCA00680
      GO TO 5                                 SCA00690
4     READ(5,105) RMAG1,RMAG2                 SCA00700
5     SET1=DATA(ISP1)                         SCA00710
      SET2=DATA(ISP2)                         SCA00720
      COEF=(RMAG2-RMAG1)/(SET2-SET1)          SCA00730
      IOP=OP                                  SCA00740
      DO 2 N=1,IDP                            SCA00750
      SNUM=(DATA(N)-SET1)*COEF+RMAG1          SCA00760
      RN=N                                    SCA00770
      FQ=RN*FQI+F8L                           SCA00780
      WRITE(2,106) SNUM                       SCA00790
106   FORMAT(E20.10)                          SCA00800
2     CONTINUE                                SCA00810
      END                                     SCA00820
```

References

1. Rubin, S., and Biehl, F. A., "Mechanical Impedance Approach to Engine Vibration Transmission into an Aircraft Fuselage," SAE Report, No. 670873, 1968.

2. Noiseux, D. U., and Meyer, E. B., "Application of Impedance Theory and Measurements to Structural Vibration," Technical Report AFFDL-TR-67-182, August 1968.

3. O'Hara, G. J., "Mechanical Impedance and Mobility Concepts," The Journal of the Acoustical Society of America, Vol. 41, No. 5, 1967, 1180-1184.

4. Schloss, F., "Recent Advances in the Measurement of Structural Impedance," SAE Report, No. 426B, 1961.

5. Wilcoxon Research, Bulletins One Through Ten, 1. "Weight Limitations of Accelerometers for High Frequency Vibration Measurements," 2. "Choosing an Impedance Head," 3. "Strain Sensitivity of Accelerometers," 4. "Inherent Limitations of Measurable Impedances," 5. "Charge Amplifiers, Disadvantages and Misconceptions," 6. "A Method of Generating Pure Lineal Motion Over an Extended Frequency Range Using Conventional Vibration Generators," 7. "Pressure Sensitivity of Accelerometers and Cables," 8. "New Small Vibration Generators for Mechanical Impedance Applications," 9. "The Effects of Electrical Termination of a Piezoelectric Transducer on its Frequency Response," 10. "Accelerometers for Model Testing."

6. Control and Communication Division, Instruction Manual, Data Acquisition System MIT Laboratory, Radiation Incorporated.

7. Spectral Dynamics Corporation of San Diego, Instruction Manual SD 1002B-33 Automatic Mechanical Impedance System, August 1968.

8. Snowdon, J. C., "Mechanical Impedance of Free-Free Beams," The Journal of the Acoustical Society of America, Vol. 37, No. 2, February 1965, 240-249.

9. Den Hartog, J. P., Mechanical Vibrations, 4th ed., McGraw-Hill Book Company, Inc., N. Y., 1956, 432.

10. Logcher, R. D.; Flachsbart, B. B.; Hall, E. J.; Power, C. M.; Wells, R. A., Jr.; Ferrante, A. J.; "Volume 1 Frame Analysis," ICES STRUDL-II The Structural Design Language Engineering User's Manual, Research Report R68-91, Structures Division and Civil Engineering Systems Laboratory, Department of Civil Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts, November 1968.

11. Logcher, R. D.; Connor, J. J., Jr.; Ferrante, A. J.; "Volume 2 Additional Design and Analysis Facilities," ICES STRUDL-II The Structural Design Language Engineering User's Manual, Research Report R68-92, Structures Division and Civil Engineering Systems Laboratory, Department of Civil Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts, June 1969.

12. Przemieniecki, J. S., Theory of Matrix Structural Analysis, McGraw-Hill Book Company, Inc., N. Y., 1968, 434.

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13. ABSTRACT

Mechanical admittance methods are applied to two simple composite structures. The mechanical admittances of the composite structures are predicted using the mechanical admittances measured on their constituent substructures. The effects of the suspension system on the measured admittances are estimated. Attention is devoted to the technique of analytically assembling substructures into a composite structure.

Resonant frequencies and mode shapes of the composite structures are estimated from the predicted admittances. The effects of modal participation and damping on the estimated mode shapes and resonant frequencies are discussed.

14. KEY WORDS

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mechanical impedance	resonant frequencies
mode shape	